Asymptotic behavior of the branching rules of classical Lie groups (after Bufetov - Gorin)

References

@ A. Bufetov, V. Gorin, Representations of classical Lie groups and quantized free convolution, 2015

… 基本的にこれの解説を目指す.

@ A. Bufetov, V. Gorin, Fluctuations of particle systems determined by Schur generating functions, 2018

… 上は branching rules の LLN. こっちは CLT

@ A. Okounkov, G. Olshanski, Asymptotics of Jack polynomials as the number of variables goes to infinity, 1998

… 漸近的表現論の基本文献

@ V. Gorin, G. Panova, Asymptotics of symmetric polynomials with applications to statistical mechanics and representation theory, 2015

… 指標の漸近解析

& 表現論, 確率論の文献を参照する

§0 Overview

- **What is a branching rule?**

G : a compact group (e.g. $U(N)$, $SO(N)$, $Sp(2N)$ ...)

$\rightsquigarrow$ $\hat{G}$: the set of all equivalence classes of irreducible representations of G

i.e., $\lambda = [\underbrace{(\pi_\lambda, V_\lambda)}_{}] \in \hat{G}$

irreducible representation $\pi_\lambda : G \curvearrowright V_\lambda$

$\rightsquigarrow$ (π, V) : a representation of G

$$\longrightarrow \quad V \cong \bigoplus_{\lambda \in \hat{G}} V_\lambda^{\oplus m_\lambda} \quad \cdots \text{ irreducible decomposition of } (\pi, V)$$

m_λ : multiplicity of (π_λ, V_λ)

Ex

- $\forall \lambda, \mu \in \hat{G}$, $\quad V_\lambda \otimes V_\mu \cong \bigoplus_{\nu \in \hat{G}} V_\nu^{\oplus c_{\lambda,\mu}^\nu}$

- $G < L$, $\quad \forall \nu \in \hat{L}$, $\quad V_\nu \cong \bigoplus_{\lambda \in \hat{G}} V_\lambda^{\oplus m_\lambda^\nu}$

$\rightsquigarrow$ **Analytic approach**: $\quad \chi := \mathrm{Tr}_V \circ \pi : G \to \mathbb{C}$: the character of G

※: $\forall \lambda \in \hat{G}$, $\chi_\lambda := \mathrm{Tr}_{V_\lambda} \circ \pi_\lambda$: the irreducible character

$$\longrightarrow \quad \chi = \sum_{\lambda \in \hat{G}} m_\lambda \chi_\lambda$$

<u>Ex</u> :

representation	tensor product	restriction
character	product	restriction

$\longrightarrow$

- $\forall \lambda, \mu \in \hat{G}, \quad \chi_\lambda \chi_\mu = \sum_{\nu \in \hat{G}} C_{\lambda\mu}^{\nu} \chi_\nu$
- $G < L, \ \forall \nu \in \hat{L}, \quad \chi_\nu|_G = \sum_{\lambda \in \hat{G}} m_\lambda^\nu \chi_\lambda$

Fourier analysis

G : a compact group $\rightsquigarrow$ $\exists$ a Haar measure on G

$\rightsquigarrow$ $\{\chi_\lambda\}_{\lambda \in \hat{G}}$: a o.n.b. of

$$L^2(G)^G := \{ f \in L^2(G) \mid \forall g \in G, \ f = f(g \cdot g^{-1}) \text{ a.e.} \}$$

$\therefore$ branching rule of G = Fourier analysis on $L^2(G)^G$

<u>Q</u> How about inductive limits of compact groups?

<u>Examples</u>

- Type A : $U(N) \longrightarrow U(N+1) \longrightarrow \cdots \longrightarrow U(\infty) = \varinjlim U(N)$

  $u \longmapsto \begin{bmatrix} u & \\ & 1 \end{bmatrix}$ $\qquad = \left\{ \begin{bmatrix} u & & \\ & 1 & \\ & & \ddots \end{bmatrix} \;\middle|\; u \in U(N) \ \forall N \right\}$

- Type B : $SO(2N+1) \to SO(2N+3) \to \cdots \to SO(2\infty+1)$

  D : $SO(2N) \to SO(2N+2) \to \cdots \to SO(2\infty)$

  $$u \mapsto \begin{bmatrix} u & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

  ※ $SO(2\infty+1) \cong SO(2\infty)$

- Type C :

  $Sp(2N) := \{ u \in U(2N) \mid J_n u J_n = -\bar{u} \}$, where $J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$

  $\rightsquigarrow \quad u = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$

  $\rightsquigarrow \quad Sp(2N) \to Sp(2N+2) \longrightarrow \cdots \to Sp(2\infty)$

  $$\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \mapsto \left(\begin{array}{cc|cc} A & & B & \\ & 1 & & 1 \\ \hline -\bar{B} & & \bar{A} & \\ & 1 & & 1 \end{array}\right)$$

Rem: $U(\infty)$, $SO(2\infty+1)$, $Sp(2\infty)$, $SO(2\infty)$ are NOT type I group

i.e., unique irreducible decomposition is not guaranteed

$\rightsquigarrow$ We consider only representations "associated with" characters

& branching rule = "decomposition" of characters

Rem: ——"—— are NOT locally compact

$\rightsquigarrow$ NO Haar measures & NO Fourier analysis

$\rightsquigarrow$ branching rules are described by probabilistic language

Def: G : a topological group (always, Hausdorff)

$f : G \to \mathbb{C}$: a continuous function

"a character of G" if
- $\forall n \geq 1, \forall g_1, \dots, g_n \in G$, $[f(g_i^{-1} g_j)]_{i,j=1}^{n} \geq 0$ (positive-semidef)
- $\forall g, h \in G$, $f(gh) = f(hg)$

$\rightsquigarrow$ $Ch(G) := \{ f \in C(G) \mid f : \text{character}, f(e) = 1 \}$

a convex set $\rightsquigarrow$ $\mathcal{E}(G) := \operatorname{ex} Ch(G)$

Rem: G : a compact group $\longrightarrow$ $\mathcal{E}(G) = \{ \frac{1}{d_\lambda} \chi_\lambda \}_{\lambda \in \widehat{G}} \cong \widehat{G}$

$(d_\lambda := \dim V_\lambda)$

Rem: GNS Construction

$\forall f \in Ch(G) \longrightarrow \exists! (\pi_f, \mathcal{H}_f, \xi_f)$

(• $(\pi_f, \mathcal{H}_f)$: a unitary representation of G

- $\xi_f \in \mathcal{H}_f$: a unit vector

s.t. $f(g) = \langle \pi_f(g)\xi_f, \xi_f \rangle \quad \forall g \in G$

What is $\Sigma(G)$ in the representation theory?

① Thm (Hirai–Hirai '05)

$\Sigma(G) \cong \{\text{finite factor representations of } G\}/\sim$

where ... $(\pi, \mathcal{H})$: a unitary representation of G

$\rightsquigarrow$ $\mathcal{R}$: the von Neumann algebra generated by $\pi(G)$

- $(\pi, \mathcal{H})$: irreducible

$\iff \mathcal{R}' := \{ T \in B(\mathcal{H}) \mid \forall S \in \mathcal{R},\ ST = TS \} = \mathbb{C}1$

- $(\pi, \mathcal{H})$: factor & finite

$\iff \mathcal{R}' \cap \mathcal{R} = \mathbb{C}1$ & $\exists$ a linear functional $\tau : \mathcal{R} \to \mathbb{C}$

s.t. $\tau \circ \pi \in Ch(G)$

② $G < G \times G$ by $G \hookrightarrow G \times G$

$g \mapsto (g, g)$

$\rightsquigarrow$ $(G \times G, G)$: a spherical pair

i.e., $(\pi, \mathcal{H})$: an irreducible representation of $G \times G$

$$\Rightarrow \quad \dim \underbrace{\mathcal{H}^G}_{:= \{ \xi \in \mathcal{H} \mid \pi(g,g)\xi = \xi \ \forall g \in G \}} \leq 1$$

Thm (Olshanski '83)

$$\Sigma(G) \cong \{ \text{irreducible spherical representations of } (G \times G, G) \} / \sim$$

where $(\pi, \mathcal{H}, \xi)$: a unitary representation of $G \times G$ with $\xi \in \mathcal{H}$

is spherical representation if ξ is cyclic & G-invariant

$\rightsquigarrow$ $g \in G \mapsto \langle \pi(g,e)\xi, \xi \rangle$ is a character

$\rightsquigarrow$ Our interests:

Branching rules of unitary representations of finite type of $G = U(\infty)$, $SO(2\infty+1)$, $Sp(2\infty)$, $SO(2\infty)$

or

spherical representations of $(G \times G, G)$

$\rightsquigarrow$ convex decomposition of $Ch(G)$

$\cdot$ $\mathrm{Ch}(G)$ ... the topology of uniform convergence of compact sets
$\cup$
$\mathcal{E}(G)$ ... the relative topology

<u>Thm</u>: $\forall f \in \mathrm{Ch}(G)$, $\exists!$ a Borel probability measure P on $\mathcal{E}(G)$
$\forall g \in G$, $$f(g) = \int_{\mathcal{E}(G)} \chi(g)\, dP(\chi)$$

$\rightsquigarrow$ P : the <u>spectral measure</u> of f

<u>Recall</u>: $G = \varinjlim G_N$, $G_N = U(N), SO(2N+1), Sp(2N), SO(2N)$
$\forall f \in \mathrm{Ch}(G)$
$\rightsquigarrow$ $f|_{G_N} \in \mathrm{Ch}(G_N)$
$\rightsquigarrow$ $\exists!$ a probability measure P_N on $\widehat{G_N} \cong \mathcal{E}(G_N)$

"<u>Thm</u>" $P_N \longrightarrow P$ weakly

$\rightsquigarrow$ Our interest : Branching rules of G or $(G \times G, G)$
= spectral measures of $\mathrm{Ch}(G)$
= asymptotic analysis of $\mathcal{P}(\widehat{G_N})$ as $N \to \infty$
prob. meas.

Plan

- Irreducible characters of the classical Lie groups
- Casimir operators of the classical Lie algebras
- Analysis of probability measures on $\mathbb{R}$
- Characters and random probability measures on $\mathbb{R}$
- Asymptotic representation theory
- Asymptotic analysis of characters

Notations

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G	G_N	$\mathfrak{g}_N$	$\hat{N}$	β_G	ε_G
A	$U(N)$	$\mathfrak{gl}_N$	N	0	0
B	$SO(2N+1)$	$\mathfrak{so}_{2N+1}$	$2N+1$	1	1
C	$Sp(2N)$	$\mathfrak{sp}_{2N}$	$2N$	-1	2
D	$SO(2N)$	$\mathfrak{so}_{2N}$	$2N$	1	0

- X : a topological space

$\rightsquigarrow$ $\mathcal{P}(X)$:= the set of Borel probability measures on X

§1 Irreducible characters of the classical Lie groups

The Peter–Weyl theorem

Fact : $\widehat{G_N} \cong \begin{cases} \{ \lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \in \mathbb{Z}^N \} & G = A . \\ \{ \lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \in \mathbb{Z}^N_{\geq 0} \} & G = B, C \\ \{ \lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \in \mathbb{Z}^{N-1}_{\geq 0} \times \mathbb{Z} \mid \lambda_{N-1} \geq |\lambda_N| \} & G = D \end{cases}$

$\rightsquigarrow$ $\forall \lambda \in \widehat{G_N}$, $\exists (\pi_\lambda, V_\lambda)$: irreducible representation of G_N

$\rightsquigarrow$ $\chi^G_\lambda := \mathrm{Tr}_{V_\lambda} \circ \pi_\lambda$: the irreducible character of G_N

$\forall f \in \mathrm{Ch}(G_N)$

$\rightsquigarrow$ By the GNS construction, $\exists ! \, (\pi_f, \mathcal{H}_f, \xi_f)$ s.t. $f = \langle \pi_f(\cdot) \xi_f, \xi_f \rangle$

$\rightsquigarrow$ irreducible decomposition : $\mathcal{H}_f \cong \bigoplus_{\lambda \in \widehat{G_N}} V_\lambda^{\oplus m_\lambda}$, $\xi_f = \sum_{\lambda \in \widehat{G_N}} \xi^\lambda$,

$\xi^\lambda \in V_\lambda^{\oplus m_\lambda}$

$$\begin{aligned} \rightsquigarrow \quad f(g) &= \langle \pi_f(g) \xi_f, \xi_f \rangle \\ &= \sum_{\lambda \in \widehat{G_N}} \langle (\pi_\lambda(g) \otimes 1_{m_\lambda}) \xi^\lambda, \xi^\lambda \rangle \longleftarrow \mathrm{End}(V_\lambda) \ni x \\ &= \sum_{\lambda \in \widehat{G_N}} \|\xi^\lambda\|^2 \cdot \frac{1}{d_\lambda} \chi_\lambda(g) \end{aligned}$$

(uniform convergence)

$\boxed{\text{trace}}$

$\mapsto \langle x \otimes 1_{m_\lambda} \cdot \xi^\lambda, \xi^\lambda \rangle$

$= \|\xi^\lambda\|^2 \frac{1}{d_\lambda} \mathrm{Tr}_{V_\lambda}(x)$

Rem: $1 = f(e) = \sum_{\lambda \in \widehat{G}_N} \|\xi^\lambda\|^2$ $\therefore$ $P(\lambda) := \|\xi^\lambda\|^2 \rightsquigarrow P \in \mathcal{P}(\widehat{G_N})$

Conversely, $\forall P \in \mathcal{P}(\widehat{G_N})$, $f = \sum_{\lambda \in \widehat{G}} P(\lambda) \frac{1}{d_\lambda} \chi_\lambda \in Ch(G_N)$

$\because$ $|\chi_\lambda(g)| \leq \chi_\lambda(e) = d_\lambda \rightsquigarrow$ (RHS) Converges uniformly

Rem: $\{\chi_\lambda\}_{\lambda \in \widehat{G_N}}$: o.n.b. of $L^2(G_N)^{G_N}$

$\rightsquigarrow f = \sum_{\lambda \in \widehat{G_N}} c_\lambda \chi_\lambda \implies c_\lambda = \langle f, \chi_\lambda \rangle$

The Weyl character formula

$\boxed{G=A}$ $\forall u \in U(N)$, $\exists v \in U(N)$ s.t. $u = v \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_N \end{pmatrix} v^{-1}$

where $z_1, \dots, z_N \in \mathbb{T} := \{ z \in \mathbb{C} \mid |z| = 1 \}$ are eigenvalues of u

$\rightsquigarrow \forall f \in Ch(U(N))$,

$$f(u) = f\left(v \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_N \end{pmatrix} v^{-1}\right)$$

$$= f\left(\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_N \end{pmatrix}\right) =: f(z_1, \dots, z_N)$$

$\boxed{G = B, C, D}$ $\qquad D(e^{i\theta}) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\leadsto \ \forall u \in G_N, \ \exists v \in G_N$

s.t $\quad u = \begin{cases} v \begin{pmatrix} D(z_1) & & \\ & \ddots & \\ & & D(z_N) \end{pmatrix} v^{-1} & G = C, D, \\ v \begin{pmatrix} D(z_1) & & & \\ & \ddots & & \\ & & D(z_N) & \\ & & & 1 \end{pmatrix} v^{-1} & G = B \end{cases}$

$\leadsto \ \forall f \in Ch(G_N), \quad f(u) = f\left(\begin{pmatrix} D(z_1) & & & \\ & \ddots & & \\ & & D(z_N) & \\ & & & (1) \end{pmatrix} \right)$

$\qquad =: f(z_1, \ldots, z_N)$

Notations :

- $\rho_{G_N} := \begin{cases} \left(\frac{N-1}{2}, \frac{N-3}{2}, \ldots, -\frac{N+1}{2}\right) & G = A \\ \left(N-\frac{1}{2}, N-\frac{3}{2}, \ldots, \frac{1}{2}\right) & G = B \\ (N, N-1, \ldots, 1) & G = C \\ (N-1, N-2, \ldots, 0) & G = D \end{cases}$

- the Weyl group

$$\underline{W_{G_N}} := \begin{cases} S_N & G = A \\ S_N \ltimes \mathbb{Z}_2^N & G = B, C \\ S_N \ltimes \mathbb{Z}_2^{N-1} & G = D \end{cases}$$

where $\mathbb{Z}_2 = \{\pm 1\}$ & $\mathbb{Z}_2^{N-1} \hookrightarrow \mathbb{Z}_2^N$

$(\varepsilon_1, \ldots, \varepsilon_{N-1}) \longmapsto (\varepsilon_1, \ldots, \varepsilon_{N-1}, \varepsilon_1 \cdots \varepsilon_{N-1})$

※ G, H : groups with $G \curvearrowright H$

$\rightsquigarrow G \ltimes H := G \times H$ with $(g, h) \cdot (g', h') := (gg', h(g \cdot h'))$

※ $\forall \sigma \in W_{A_N} = S_N$, $\underline{\mathrm{sgn}(\sigma) := (-1)^{\mathrm{inv}(\sigma)}}$,

where $\mathrm{inv}(\sigma) := \{ (i, j) \mid i < j, \ \sigma(i) > \sigma(j) \}$

$\forall w = (\sigma, (\varepsilon_1, \ldots, \varepsilon_N)) \in W_{G_N}$ $(G \neq A)$,

$\underline{\mathrm{sgn}(w) := \varepsilon_1 \cdots \varepsilon_N \, \mathrm{sgn}(\sigma)}$

<u>Thm</u> (the Weyl character formula)

$\forall \lambda \in \hat{G}_N$,

$$\chi_\lambda^G (z_1, \ldots, z_N) = \frac{\sum_{w \in W_{G_N}} \mathrm{sgn}(w) \, z^{w \cdot (\lambda + \rho)}}{\sum_{w \in W_{G_N}} \mathrm{sgn}(w) \, z^{w \cdot \rho}}$$

where $z^\alpha := z_1^{\alpha_1} \cdots z_N^{\alpha_N}$ for $\alpha = (\alpha_1, \ldots, \alpha_N)$

$$w \cdot \alpha = (\varepsilon_1 \alpha_{\sigma(1)}, \ldots, \varepsilon_N \alpha_{\sigma(N)})$$

Ex

- $$\chi^A_\lambda(z_1, \ldots, z_N) = \frac{\det[z_i^{\lambda_j+N-j}]_{i,j=1}^N}{\det[z_i^{N-j}]_{i,j=1}^N}$$ … the Schur polynomial

- $$\chi^G_\lambda(z_1, \ldots, z_N) = \frac{\det[z_i^{\lambda_j+N-j+\varepsilon_G/2} - z_i^{-(\lambda_j+N-j+\varepsilon_G/2)}]_{i,j=1}^N}{\prod_{i=1}^N (z_i^{\varepsilon_G/2} - z_i^{-\varepsilon_G/2}) \prod_{1\le i<j\le N} (z_i + z_i^{-1} - (z_j + z_j^{-1}))}$$

  $(G = B, C)$

  where $\varepsilon_G = \begin{cases} 1 & G = B \\ 2 & G = C \end{cases}$

- $$\chi^D_\lambda(z_1, \ldots, z_N) = \frac{\det[z_i^{\lambda_j+N-j} - z_i^{-(\lambda_j+N-j)}]_{i,j=1}^N + \det[z_i^{\lambda_j+N-j} - z_i^{-(\lambda_j+N-j)}]_{i,j=1}^N}{\prod_{1\le i<j\le N} (z_i + z_i^{-1} - (z_j + z_j^{-1}))}$$

Rem $\underset{(1,\ldots,1)}{\chi^G_\lambda(1^N)}$ = the dimension of the irreducible representation with label λ $\rightsquigarrow$ d^G_λ

$\rightsquigarrow$ the Weyl dimension formula

- $$d^A_\lambda = \prod_{1\le i<j\le N} \frac{\lambda_i - i - \lambda_j + j}{j - i}$$

- $$d_\lambda^G = d_\lambda^A \prod_{1\le i<j\le N} \frac{\lambda_i+\lambda_j+2N-i-j+\varepsilon_G}{2N-i-j+\varepsilon_G} \prod_{i=1}^{N} \frac{\lambda_i+N-i+\varepsilon_G/2}{N-i+\varepsilon_G/2}$$
  $(G=B,C)$

- $$d_\lambda^D = d_\lambda^A \prod_{1\le i<j\le N} \frac{\lambda_i+\lambda_j+2N-i-j}{2N-i-j}$$

§2 Casimir operators of the Classical Lie algebras

$\mathfrak{g}_N :=$ the Complexified Lie algebra of G_N

$$= \begin{cases} \mathfrak{gl}_N & G = A \\ \mathfrak{so}_{2N+1} & G = B \\ \mathfrak{sp}_{2N} & G = C \\ \mathfrak{so}_{2N} & G = D \end{cases}$$

<u>Def</u> • $\mathfrak{gl}_N :=$ the Lie algebra generated by E_{ij} $(i,j = 1, \dots, N)$

s.t. $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$

• $I_{G_N} := \begin{cases} \{1 < 2 < \cdots < N\} & G = A \\ \{1 < \cdots < N < 0 < -N < \cdots < -1\} & G = B \\ \{1 < \cdots < N < -N < \cdots < -1\} & G = C, D \end{cases}$

ordered index set

$\rightsquigarrow$ $\hat{N} := |I_{G_N}|$,

$\forall\, i \in I_{G_N}$, $\bar{i} := -i$

$\rightsquigarrow$ $\mathfrak{so}_{2N+1 \text{ or } 2N} :=$ the Lie subalgebra of $\mathfrak{gl}_{I_{G_N}}$ $(G = B, D)$

generated by $X_{ij} := E_{ij} - E_{\bar{j}\bar{i}}$ $(i, j \in I_{G_N})$

Rem $\quad X_{ij} = -X_{\bar{j}\bar{i}}$

& $\mathfrak{sp}_{2N}$:= the Lie subalgebra of $\mathfrak{gl}_{I_{CN}}$
generated by $X_{ij} := E_{ij} - \varepsilon_i \varepsilon_j E_{\bar{j}\bar{i}} \quad (i, j \in I_{CN})$
where $\varepsilon_i := \mathrm{sgn}(i)$

Rem $\quad X_{ij} = -\varepsilon_i \varepsilon_j X_{\bar{j}\bar{i}}$

Def: $U(\mathfrak{g}_N)$:= the universal enveloping algebra of $\mathfrak{g}_N$
i.e., the universal $\mathbb{C}$-algebra generated by $\mathfrak{g}_N$
s.t. $\forall X, Y \in \mathfrak{g}_N, \quad XY - YX = [X, Y]$

$\rightsquigarrow Z(\mathfrak{g}_N) := \{ Z \in U(\mathfrak{g}_N) \mid ZX = XZ \quad \forall X \in U(\mathfrak{g}_N) \}$
a Casimir operator

Later, we will study the asymptotic behavior of "spectral measures" of Casimir operators

- The Work of Perelmov - Popov

Fix a finite-dimensional representation of $\mathfrak{g}_N$: $X_\alpha \mapsto A_\alpha \quad (\alpha \in I_{GN})$
$\rightsquigarrow \{A^\alpha\}_{\alpha \in I_{GN}}$ s.t. $\mathrm{Tr}(A^\alpha A_\beta) = \delta_{\alpha, \beta}$

※ $I \subset I_{G_N}$ s.t. $\{X_\alpha\}_{\alpha \in I}$: a basis of $\mathfrak{g}_N$

$\rightsquigarrow$ $\hat{g}_{\alpha,\beta} := \mathrm{Tr}(A_\alpha A_\beta)$

& $\hat{g} := (\hat{g}_{\alpha\beta})_{\alpha,\beta \in I}$: assumed to be invertible

$\rightsquigarrow$ $\hat{g}^{-1} = (\hat{g}^{\alpha,\beta})_{\alpha,\beta \in I}$ & $A^\alpha := \sum_{\beta \in I} \hat{g}^{\alpha\beta} A_\beta$

$\rightsquigarrow$ $\mathrm{Tr}(A^\alpha A_\beta) = \sum_\gamma \hat{g}^{\alpha\gamma} \hat{g}_{\gamma\beta} = \delta_{\alpha,\beta}$

<u>Rem</u> $\forall \alpha, \beta \in I$, $[X_\alpha, X_\beta] = \sum_{\gamma \in I} C_{\alpha,\beta}^{\gamma} X_\gamma$

$$\begin{aligned} \rightsquigarrow\quad 0 &= \mathrm{Tr}([A_\alpha, A_\beta A_\gamma]) \\ &= \mathrm{Tr}([A_\alpha, A_\beta] A_\gamma) + \mathrm{Tr}(A_\beta [A_\alpha, A_\gamma]) \\ &= \sum_\delta C_{\alpha\beta}^{\delta} \hat{g}_{\delta\gamma} + \sum_\delta \hat{g}_{\beta\delta} C_{\alpha\gamma}^{\delta} \end{aligned}$$

$$\therefore \quad \hat{g}\, C_\alpha = - C_\alpha^T \hat{g} \qquad (2.1)$$

where $C_\alpha := (C_{\alpha,\gamma}^{\delta})_{\delta,\gamma \in I}$ i.e., $C_\alpha^T = (C_{\alpha,\delta}^{\gamma})_{\delta,\gamma \in I}$

<u>Def</u> $\forall p \geq 1$, $\forall \alpha_1, \ldots, \alpha_p \in I_{G_N}$

- $g^{\alpha_1, \ldots, \alpha_p} = \mathrm{Tr}(A^{\alpha_1} \cdots A^{\alpha_p})$

$$\bullet \quad C_p := \sum_{\alpha_1,\dots,\alpha_p \in I_{G_N}} g^{\alpha_1,\dots,\alpha_p} X_{\alpha_1} \cdots X_{\alpha_p} \in U(\mathfrak{g}_N)$$

$$= (\mathrm{Tr} \otimes \mathrm{id})(\tilde{A}^p), \quad \text{where } \tilde{A} = \sum_{\alpha \in I_{G_N}} A^\alpha \otimes X_\alpha$$

$$\bullet \quad (T_p)^\alpha = \sum_{\alpha_1,\dots,\alpha_p \in I_{G_N}} g^{\alpha_1,\dots,\alpha_p,\alpha} X_{\alpha_1} \cdots X_{\alpha_p}$$

$$= (\mathrm{Tr} \otimes \mathrm{id})(\tilde{A}^p \cdot A^\alpha \otimes 1)$$

$$\rightsquigarrow \quad C_p = \sum_{\alpha \in I_{G_N}} (\mathrm{Tr} \otimes \mathrm{id})(\tilde{A}^{p-1} \, A^\alpha \otimes X_\alpha)$$

$$= \sum_{\alpha \in I_{G_N}} (T_{p-1})^\alpha X_\alpha$$

<u>Lemma</u> $\forall \alpha, \beta \in I_{G_N}$, $\quad [X_\alpha, (T_p)^\beta] = \sum_\gamma \tilde{C}^\gamma_{\alpha,\beta} (T_p)^\gamma$,

where $\tilde{C}_\alpha = -C_\alpha^T \qquad -C^\beta_{\alpha,\gamma}$

proof) $\quad [X_\alpha, (T_p)^\beta]$

$$= \sum_{\beta_1,\dots,\beta_p} g^{\beta_1,\dots,\beta_p,\beta} [X_\alpha, X_{\beta_1} \cdots X_{\beta_p}]$$

$$= \sum_{\beta_1,\dots,\beta_p} g^{\beta_1,\dots,\beta_p,\beta} \sum_{j=1}^{p} X_{\beta_1} \cdots X_{\beta_{j-1}} [X_\alpha, X_{\beta_j}] X_{\beta_{j+1}} \cdots X_{\beta_p}$$

$$= \sum_{\beta_1,\dots,\beta_p} g^{\beta_1,\dots,\beta_p,\beta} \sum_{j=1}^{p} \sum_{\gamma} C^{\gamma}_{\alpha,\beta_j} X_{\beta_1} \cdots X_{\gamma} \cdots X_{\beta_p}$$

$$= \sum_{j=1}^{p} \sum_{\substack{\beta_1,\dots,\beta_p,\\ \gamma}} g^{\beta_1,\dots,\gamma,\dots,\beta_p,\beta} C^{\beta_j}_{\alpha,\gamma} X_{\beta_1} \cdots X_{\beta_p}$$

$$\rightsquigarrow \sum_{j=1}^{p} \sum_{\gamma} g^{\beta_1,\dots,\gamma,\dots,\beta_p,\beta} C^{\beta_j}_{\alpha,\gamma}$$

$$= \sum_{j=1}^{p} \mathrm{Tr}\left(A^{\beta_1} \cdots \left(\sum_{\gamma} C^{\beta_j}_{\alpha,\gamma} A^{\gamma} \right) \cdots A^{\beta_p} A^{\beta} \right)$$

$$\because \sum_{\gamma} C^{\beta_j}_{\alpha,\gamma} A^{\gamma} = \sum_{\gamma} \sum_{\delta} C^{\beta_j}_{\alpha,\gamma} \hat{g}^{\gamma,\delta} A_{\delta}$$

$$= \sum_{\delta} \left(C_{\alpha} \hat{g}^{-1} \right)_{\beta_j,\delta} A_{\delta}$$

$$= - \sum_{\delta} \left(\hat{g}^{-1} C_{\alpha}^{T} \right)_{\beta_j,\delta} A_{\delta} \qquad (2.1)$$

$$= - \sum_{\delta,\gamma} \hat{g}^{\beta_j \gamma} C^{\delta}_{\alpha\gamma} A_{\delta}$$

$$= - \sum_{\gamma} \hat{g}^{\beta_j,\gamma} [A_{\alpha}, A_{\gamma}]$$

$$= - [A_{\alpha}, A^{\beta_j}]$$

$$= - \sum_{j=1}^{p} \mathrm{Tr}\left(A^{\beta_1} \cdots [A_{\alpha}, A^{\beta_j}] \cdots A^{\beta_p} A^{\beta} \right)$$

$$= -\operatorname{Tr}\left([A_\alpha, A^{\beta_1} \cdots A^{\beta_j}] A^\beta \right)$$

$$= \operatorname{Tr}\left(A^{\beta_1} \cdots A^{\beta_p} [A_\alpha, A^\beta] \right)$$

$$\therefore \quad [X_\alpha, (T_p)^\beta] = \sum_{\beta_1, \dots, \beta_p} \operatorname{Tr}\left(A^{\beta_1} \cdots A^{\beta_p} [A_\alpha, A^\beta] \right) X_{\beta_1} \cdots X_{\beta_p}$$

$$\because \quad [A_\alpha, A^\beta] = \sum_{\gamma, \delta} \hat{g}^{\beta\gamma} C^{\delta}_{\alpha\gamma} A_\delta$$

$$= \sum_{\delta} (C_\alpha \hat{g}^{-1})_{\delta\beta} A_\delta \tag{2.1}$$

$$= \sum_{\delta} (\hat{g}^{-1} \tilde{C}_\alpha)_{\delta\beta} A_\delta$$

$$= \sum_{\delta, \gamma} \hat{g}^{\delta\gamma} \tilde{C}^{\gamma}_{\alpha\beta} A_\delta$$

$$= \sum_{\gamma} \tilde{C}^{\gamma}_{\alpha\beta} A^\gamma$$

$$\rightsquigarrow \quad [X_\alpha, (T_p)^\beta] = \sum_{\beta_1, \dots, \beta_p} \sum_{\gamma} \tilde{C}^{\gamma}_{\alpha\beta} \, g^{\beta_1, \dots, \beta_p, \gamma} X_{\beta_1} \cdots X_{\beta_p}$$

$$= \sum_{\gamma} \tilde{C}^{\gamma}_{\alpha\beta} (T_p)^\gamma \qquad \square$$

<u>Prop</u> $\quad C_p \in Z(\mathfrak{g}_N)$

proof) $\quad \forall \alpha \in I_{G_N}, \quad [X_\alpha, C_p] = \sum_\beta [X_\alpha, (T_{p-1})^\beta X_\beta]$

$$= \sum_\beta [X_\alpha, (T_{p-1})^\beta] X_\beta + (T_{p-1})^\beta [X_\alpha, X_\beta]$$

$$= \sum_\beta \sum_\gamma \tilde{C}_{\alpha,\beta}^{\gamma} (T_{p-1})^\gamma X_\beta + C_{\alpha,\beta}^{\gamma} (T_{p-1})^\beta X_\gamma$$

$$= \sum_{\beta,\gamma} \underbrace{(\tilde{C}_{\alpha,\beta}^{\gamma} + C_{\alpha,\gamma}^{\beta})}_{=0} (T_{p-1})^\gamma X_\beta = 0$$

<u>Examples</u>

- $\mathfrak{g}_N = \mathfrak{gl}_N$ & the dual fundamental representation $E_{ij} \longmapsto -e_{ji}$ on $\mathbb{C}^N$

$\rightsquigarrow \quad e^{ij} = -e_{ij} \quad \cdots \quad \mathrm{Tr}(e^{ij}(-e_{lk})) = \mathrm{Tr}(e_{ij}e_{lk}) = \delta_{(i,j),(k,l)}$

$$\rightsquigarrow \quad C_p = \sum_{\substack{i_1,\dots,i_p \\ j_1,\dots,j_p}} \mathrm{Tr}(e^{i_1 j_1} \cdots e^{i_p j_p}) E_{i_1 j_1} \cdots E_{i_p j_p}$$

$$= (-1)^p \sum_{\substack{i_1,\dots,i_p \\ j_1,\dots,j_p}} \mathrm{Tr}(e_{i_1 j_1} \cdots e_{i_p j_p}) E_{i_1 j_1} \cdots E_{i_p j_p}$$

$$= (-1)^p \sum_{i_1,\dots,i_p} E_{i_1 i_2} \cdots E_{i_p i_1}$$

@ $\mathfrak{g}_N = \mathfrak{so}_{2N+1},\ \mathfrak{sp}_{2N}$

$$X_{ij} = E_{ij} - E_{\bar{j}\bar{i}} \longmapsto x_{ij} = -(e_{ji} - e_{\bar{i}\bar{j}})$$

$$\rightsquigarrow \quad x^{ij} = x_{ji}/2 \quad \cdots \quad (i,j) \neq (\bar{l},\bar{k}) \Rightarrow \mathrm{Tr}(x^{ij} x_{kl}) = \delta_{(i,j),(k,l)}$$

$$\rightsquigarrow (\hat{A})_{ij} = \sum_{k,l} (x^{kl})_{ij} X_{kl}$$

$$= \frac{1}{2} \sum_{k,l} (x_{lk})_{ij} X_{kl} \qquad -(e_{kl} - e_{\bar{l}\bar{k}})_{ij}$$

$$= -\frac{1}{2}(X_{ij} - X_{\bar{j}\bar{i}}) = -X_{ij}$$

$$\rightsquigarrow \quad C_p = (\mathrm{Tr} \otimes \mathrm{id})(\tilde{A}^p)$$

$$= \sum_{i_1,\dots,i_p} (\hat{A})_{i_1 i_2} \cdots (\hat{A})_{i_p i_1}$$

$$= (-1)^p \sum_{i_1,\dots,i_p} X_{i_1 i_2} \cdots X_{i_p i_1}$$

@ $\mathfrak{g}_N = \mathfrak{sp}_{2N}$

$$X_{ij} = E_{ij} - \varepsilon_i \varepsilon_j E_{\bar{j}\bar{i}} \longmapsto x_{ij} = -(e_{ji} - \varepsilon_i \varepsilon_j e_{\bar{i}\bar{j}})$$

$$\rightarrow \quad x^{ij} = x_{ji}/2 \quad \cdots \quad (i,j) \neq (\bar{l},\bar{k}) \Rightarrow \mathrm{Tr}(x^{ij} x_{kl}) = \delta_{(i,j),(k,l)}$$

$$\rightsquigarrow \quad (\widehat{A})_{ij} = \sum_{k,l} (x^{kl})_{ij} \, X_{kl}$$

$$= \frac{1}{2} \sum_{k,l} (x_{lk})_{ij} \, X_{kl}$$

$$= -\frac{1}{2} \left(X_{ij} - \varepsilon_i \varepsilon_j X_{\bar{j}\bar{i}} \right) = -X_{ij}$$

$$\rightsquigarrow \quad C_p = \sum_{i_1, \dots, i_p} (\widehat{A})_{i_1 i_2} \cdots (\widehat{A})_{i_p i_1}$$

$$= (-1)^p \sum_{i_1, \dots, i_p} X_{i_1 i_2} \cdots X_{i_p i_1}$$

<u>Thm</u>: $G = A, B, C, D$,

$\forall p \geq 1$, $\quad \widehat{C}_p^G := \sum_{i_1, \dots, i_p} X_{i_1 i_2} X_{i_2 i_3} \cdots X_{i_p i_1} \in Z(\mathfrak{g}_N)$

where $\quad X_{ij} = \begin{cases} E_{ij} & G = A \\ E_{ij} - E_{\bar{j}\bar{i}} & G = B, D \\ E_{ij} - \varepsilon_i \varepsilon_j E_{\bar{j}\bar{i}} & G = C \end{cases}$

<u>Rem</u>: $\quad (T^{p-1})_{ij} = \sum_{i_2, \dots, i_{p-1}} X_{i i_1} X_{i_1 i_2} \cdots X_{i_{p-1} j}$

$$\rightsquigarrow \quad \widehat{C}_p^G = \sum_{i,j} (T^{p-1})_{ij} \, X_{ji}$$

◎ Eigenvalues of $\widehat{C}_p^G$

Rem : $\forall \lambda \in \widehat{G}_N$, (π_λ, V_λ) : the irreducible representation of $\mathfrak{g}_N$

$$\rightsquigarrow \quad \pi_\lambda(\widehat{C}_p^G) = \exists\, C_p^G(\lambda) \cdot 1_{V_\lambda}$$

Thm 2.1
$$C_p^G(\lambda) = \sum_{i \in I_{G_N}} \Lambda_i^p \, \xi_i \,,$$

where

$$\Lambda_i = \lambda_i + N - i \quad \text{if } G = A$$

$$\Lambda_i = \begin{cases} \lambda_i + 2N - i & (i = 1, \dots, N) \\ N & (i = 0) \\ -\lambda_{-i} - i - 1 & (i = -N, \dots, -1) \end{cases} \quad \text{if } G = B$$

$$\Lambda_i = \begin{cases} \lambda_i + 2N - i + 1 & (i = 1, \dots, N) \\ -\lambda_{-i} - i + 1 & (i = -N, \dots, -1) \end{cases} \quad \text{if } G = C$$

$$\Lambda_i = \begin{cases} \lambda_i + 2N - i - 1 & (i = 1, \dots, N) \\ -\lambda_{-i} - i - 1 & (i = -N, \dots, -1) \end{cases} \quad \text{if } G = D$$

&

$$\xi_i = \prod_{j \neq i} \left(1 - \frac{1 - \beta_G \delta_{-i\,j}}{\Lambda_i - \Lambda_j} \right), \qquad \beta_G = \begin{cases} 0 & G = A \\ 1 & G = B, D \\ -1 & G = C \end{cases}$$

Assume X_{ij} $(i<j)$ $\longleftrightarrow$ positive roots

<u>Lem 2.2</u> $i<j \Rightarrow (\hat{T}^P)_{ij}\psi_\lambda = 0$

proof) $\forall k=1,\dots,N$,

$$[X_{kk}, (\hat{T}^P)_{ij}] = \sum_{s,t} \tilde{C}^{(s,t)}_{(k,k),(i,j)} (\hat{T}^P)_{st}$$

$$= -\sum_{s,t} C^{(i,j)}_{(k,k),(s,t)} (\hat{T}^P)_{st}$$

※ $s\neq t \Rightarrow [X_{kk}, X_{st}] = (\delta_{s,k} - \delta_{t,k}) X_{st}$

$$\therefore\quad C^{(i,j)}_{(k,k),(s,t)} = \begin{cases} \delta_{s,k} & (i,j)=(s,t), \\ -\delta_{t,k} & (i,j)=(s,t), \\ 0 & \text{otherwise} \end{cases}$$

$[X_{kk}, X_{ss}] = 0 \quad \therefore\ C^{(i,j)}_{(k,k),(s,s)} = 0$

$$\rightsquigarrow [X_{kk}, (\hat{T}^P)_{ij}] = -(\delta_{i,k} - \delta_{j,k}) (\hat{T}^P)_{ij}$$

$$\therefore\quad X_{kk} (\hat{T}^P)_{ij}\psi_\lambda = \{ [X_{kk}, (\hat{T}^P)_{ij}] + (\hat{T}^P)_{ij} X_{kk} \}\psi_\lambda$$

$$= (\lambda_i + \underbrace{(\delta_{j,k} - \delta_{i,k})}_{\text{positive root if } j>i}) (\hat{T}^P)_{ij}\psi_\lambda$$

$\therefore\quad (\hat{T}^P)_{ij}\psi_\lambda = 0$ if $j>i$

$\blacksquare$

$\underline{\text{Lem 2.3}}$ $\quad C_p(\lambda) = \sum_{i,j \in I_{G_N}} (a^p)_{ij}$, where $a_{ij} = \Lambda_i \delta_{i,j} - (1 - \beta_G \delta_{\bar{i},j}) \delta_{j>i}$

Proof) $\quad \hat{C}_p \psi_\lambda = \sum_{i,j \in I_{G_N}} (\hat{T}^{p-1})_{ij} X_{ji} \psi_\lambda$

$$= \sum_i \varepsilon_i \lambda_i (\hat{T}^{p-1})_{ii} \psi_\lambda + \sum_{j>i} (\hat{T}^{p-1})_{ij} X_{ji} \psi_\lambda$$

Lem 2.2

$$= \sum_i \varepsilon_i \lambda_i (\hat{T}^{p-1})_{ii} \psi_\lambda + \sum_{j>i} [(\hat{T}^{p-1})_{ij}, X_{ji}] \psi_\lambda$$

※) $[(\hat{T}^q)_{ij}, X_{ji}] = (1 - \beta_G \delta_{\bar{i},j}) ((\hat{T}^q)_{ii} - (\hat{T}^q)_{jj})$

∵) $(\hat{T}^q)_{ij} = \sum_k (\hat{T}^{q-1})_{ik} X_{kj} \rightsquigarrow$ Induction on q //

$$\rightsquigarrow \quad = \sum_i \varepsilon_i \lambda_i (\hat{T}^{p-1})_{ii} \psi_\lambda + \sum_{j>i} (1 - \beta_G \delta_{\bar{i},j}) ((\hat{T}^{p-1})_{ii} - (\hat{T}^{p-1})_{jj}) \psi_\lambda$$

$\boxed{G = A}$ $\quad \hat{C}_p \psi_\lambda = \sum_{i=1}^N \lambda_i (\hat{T}^{p-1})_{ii} \psi_\lambda + \sum_{1 \le i < j \le N} ((\hat{T}^{p-1})_{ii} - (\hat{T}^{p-1})_{jj}) \psi_\lambda$

$$= \sum_{i=1}^N (\lambda_i + (N-i) - (i-1)) (\hat{T}^{p-1})_{ii} \psi_\lambda$$

$$= \sum_{i=1}^N (\lambda_i + N - 2i + 1) (\hat{T}^{p-1})_{ii} \psi_\lambda$$

& $\forall q \geq 1$,

$$(\hat{T}^{q})_{ii}\psi_\lambda = (\hat{T}^{q-1})_{ii} X_{ii}\psi_\lambda + \sum_{j>i} (\hat{T}^{q-1})_{ij} X_{ji}\psi_\lambda$$

$$= \lambda_i (\hat{T}^{q-1})_{ii}\psi_\lambda + \sum_{j>i} \left((\hat{T}^{q-1})_{ii} - (\hat{T}^{q-1})_{jj}\right)\psi_\lambda$$

$$= (\lambda_i + N - i)(\hat{T}^{q-1})_{ii}\psi_\lambda - \sum_{j>i} (\hat{T}^{q-1})_{jj}\psi_\lambda$$

$$= \sum_{j} a_{ij} (\hat{T}^{q-1})_{jj}\psi_\lambda$$

$$= \sum_{j} (a^{q-1})_{ij} (\hat{T})_{jj}\psi_\lambda \qquad (\hat{T})_{jj} = X_{jj}$$

$$= \sum_{j} (a^{q-1})_{ij}\cdot \lambda_j \psi_\lambda$$

$$\therefore \quad \hat{C}_p\psi_\lambda = \sum_i (\lambda_i + N - 2i + 1)(\hat{T}^{p-1})_{ii}\psi_\lambda$$

$$= \sum_{i,j} (\lambda_i + N - 2i + 1)(a^{p-2})_{ij}\lambda_j\psi_\lambda$$

$$\because \quad a_{ij} = \Lambda_i\delta_{ij} - \delta_{j>i} \quad \rightsquigarrow \quad \sum_j a_{ij} = \Lambda_i - (N-i) = \lambda_i$$

$$\Lambda_i = \lambda_i + N - i$$

$$\sum_i a_{ij} = \Lambda_j - (j-1) = \lambda_j + N - 2j + 1$$

$$\rightsquigarrow \quad = \sum_{k,i,j,l} a_{ki}(a^{p-2})_{ij} a_{jl}\psi_\lambda = \sum_{k,l} (a^p)_{kl}\psi_\lambda \qquad \blacksquare$$

Proof of Thm 2.1 in type A)

$$a = \begin{pmatrix} \Lambda_1 & & -1 \\ & \ddots & \\ 0 & & \Lambda_N \end{pmatrix} = V \begin{pmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_N \end{pmatrix} U$$

where $V = (v^{(1)} \cdots v^{(N)})$: left eigenvectors

$$U = \begin{pmatrix} {}^t u^{(1)} \\ \vdots \\ {}^t u^{(N)} \end{pmatrix}$$: right eigenvectors

s.t. $\langle v^{(i)}, u^{(j)} \rangle = \delta_{i,j}$

i.e., $a = \sum_i \Lambda_i \, u^{(i)} \cdot {}^t v^{(i)}$

$$\rightsquigarrow \quad v^{(i)}_j = \begin{cases} -\dfrac{1}{\Lambda_i - \Lambda_j} \displaystyle\prod_{k=j+1}^{i-1} \left(1 - \dfrac{1}{\Lambda_i - \Lambda_k}\right) & j = 1, \ldots, i-1 \\ 1 & j = i \\ 0 & j = i+1, \ldots, N \end{cases}$$

$$u^{(i)}_j = \begin{cases} 0 & j = 1, \ldots, i-1, \\ 1 & j = i \\ -\dfrac{1}{\Lambda_i - \Lambda_j} \displaystyle\prod_{k=i+1}^{N} \left(1 - \dfrac{1}{\Lambda_i - \Lambda_k}\right) & j = i+1, \ldots, N \end{cases}$$

$\therefore$ By Lem 2.3,

$$C_p(\lambda) = \left\langle a^p \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle$$

$$= \sum_{i=1}^{N} \Lambda_i^p \left\langle u^{(i)} \otimes v^{(i)} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle$$

$$= \sum_{i=1}^{N} \Lambda_i^p \left\langle u^{(i)}, \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle \left\langle v^{(i)}, \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle$$

$$= \sum_{i=1}^{N} \Lambda_i^p \xi_i$$

$\blacksquare$

<u>Thm</u> $\forall \lambda \in \hat{G}_N$,

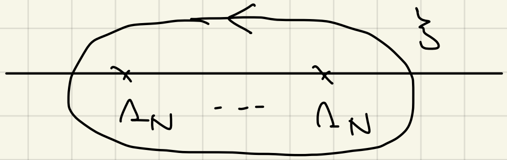
$$C_\lambda(z) := \sum_{p=0}^{\infty} C_p(\lambda) z^p = \left(1 + \frac{\beta_G z}{2 - (2N + \varepsilon_G - 1)z}\right) \frac{1 - \Pi_\lambda(z)}{z}$$

where $\displaystyle \Pi_\lambda(z) := \prod_{i \in I_{G_N}} \left(1 - \frac{z}{1 - \Lambda_i z}\right)$

$$\beta_G = \begin{cases} 0 & G = A, \\ 1 & G = B, D \\ -1 & G = C \end{cases}, \qquad \varepsilon_G = \begin{cases} 1 & G = B \\ 2 & G = C \\ 0 & G = D \end{cases}$$

Proof in type A)

By Thm 2.1, $\displaystyle C_p(\lambda) = \sum_i \Lambda_i^p \prod_{j \neq i} \left(1 - \frac{1}{\Lambda_i - \Lambda_j}\right)$



$$= \frac{1}{2\pi i} \oint \zeta^{p} \prod_{j=1}^{N} \left(1 - \frac{1}{\zeta - \Lambda_j}\right) d\zeta$$

$w = \zeta^{-1}$

$$= \frac{1}{2\pi i} \oint \frac{1}{w^{p}} \underbrace{\prod_{j=1}^{N} \left(1 - \frac{w}{1 - \Lambda_j w}\right)} \left(-\frac{1}{w^2}\right) dw$$

$\Pi_\lambda(w)$: holomorphic inside of the Contour

$$\therefore \quad C_\lambda(z) = \sum_{p=0}^{\infty} C_p(\lambda) z^p$$

$$= \frac{-1}{2\pi i} \oint \frac{1}{w^2} \cdot \frac{1}{1 - z/w} \cdot \Pi_\lambda(w)\, dw$$

$$= \frac{-1}{2\pi i} \oint \frac{1}{w} \frac{1}{w - z} \Pi_\lambda(w)\, dw$$

$$= -\left(\frac{1}{-z} \Pi_\lambda(0) + \frac{1}{z} \Pi_\lambda(z) \right)$$

$$= \frac{1 - \Pi_\lambda(z)}{z}$$

∎

§3 Analysis of probability measures on $\mathbb{R}$

$\mathcal{P}(\mathbb{R})$:= the set of all Borel probability measures on $\mathbb{R}$

Def: $m_n \in \mathcal{P}(\mathbb{R})$ $(n \geq 1)$, $m \in \mathcal{P}(\mathbb{R})$

- $m_n \to m$ weakly as $n \to \infty$ if $\forall f \in C(\mathbb{R})$: bounded,
$$\int_{\mathbb{R}} f(x)\,dm_n(x) \longrightarrow \int_{\mathbb{R}} f(x)\,dm(x)$$

- $m_n \to m$ in the moment sense as $n \to \infty$ if $\forall p \geq 1$,
$$\int_{\mathbb{R}} x^p\,dm_n(x) \longrightarrow \int_{\mathbb{R}} x^p\,dm(x)$$

Lem $\exists r > 0$ s.t. $\mathrm{Supp}(m_n) \subset [-r, r]$ $\forall n$

$\Rightarrow$ $m_n \to m$ weakly $\Leftrightarrow$ $m_n \to m$ in the moment sense

Then, $\mathrm{Supp}(m) \subset [-r, r]$

Def $m \in \mathcal{P}(\mathbb{R})$

$\rightsquigarrow$ The Cauchy transform of m
$$G_m(z) := \int_{\mathbb{R}} \frac{1}{z - x}\,dm(x) \quad \forall z \in \mathbb{C} \setminus \mathrm{Supp}(m)$$

<u>Rem</u>. $\operatorname{Im}(z) > 0 \implies \left|\frac{1}{z-x}\right| \leq \frac{1}{\operatorname{Im} z}$

$\therefore$ $m_n \to m$ weakly as $n \to \infty$ $\implies$ $G_{m_n}(z) \xrightarrow[n\to\infty]{} G_m(z)$ (pt-wise)

<u>Thm</u>: $m_n \in \mathcal{P}(\mathbb{R})$ $(n \geq 1)$

Assume
- $\exists\, G(z) = \lim_{n\to\infty} G_{m_n}(z) \quad \forall z \in \mathbb{C}^+$
- $\lim_{y\to\infty} iy\, G(iy) = 1$

Then $\exists!\, m \in \mathcal{P}(\mathbb{R})$ s.t. $m_n \to m$ weakly as $n \to \infty$

$G_m(z) = G(z)$

<u>Rem</u>:
- G : analytic function from $\mathbb{C}^+$ to $\mathbb{C}^-$
- $\lim_{y\to\infty} iy\, G(iy) = 1$

$\implies$ $\exists!\, m \in \mathcal{P}(\mathbb{R})$ s.t. $G(z) = G_m(z)$

"Proof" $\forall z \in \mathbb{C}^+$, $|G_{m_n}(z)| \leq \operatorname{Im}(z)^{-1}$

$\rightsquigarrow$ $\{G_{m_n}(z)\}_{n\geq 1}$: uniformly bounded on a compact set of $\mathbb{C}^+$

$\rightsquigarrow$ by Montel's thm, $\exists$ subseq $G_{m_{n_k}}(z) \to G(z)$ uniformly on compact sets

$\therefore$ G : analytic & $\mathrm{Im}\, G(z) \leq 0$ $\forall z \in \mathbb{C}^+$

Moreover, $\mathrm{Im}\, G(z) < 0$!

$\therefore$ $\exists!\ m \in \mathcal{P}(\mathbb{R})$ s.t. $G_m(z) = G(z)$

On the other hand, by the Banach-Alaoglu thm,

$\exists$ subseq $m_{n_k'} \longrightarrow {}^{\exists}\nu \in \mathcal{B}(\mathbb{R})$ in $C_0(\mathbb{R})^*$

$\therefore$ $x \mapsto \dfrac{1}{z-x} \in C_0(\mathbb{R})$ $\rightsquigarrow$ $G_{m_{n_k'}}(z) \longrightarrow G_\nu(z)$

$\rightsquigarrow$ $G_\nu = G_m$ $\therefore$ $\nu = m$

$\rightsquigarrow$ Since $m_{n_k'} \in \mathcal{P}(\mathbb{R})$, $m_{n_k'} \longrightarrow \nu = m$ weakly

i.e., all weak cluster pt $= m$

$\therefore$ $m_n \longrightarrow m$ weakly $\square$

Lem: $m \in \mathcal{P}(\mathbb{R})$: compact support & $C_m(z) := G_m(z^{-1})$

$\exists$ analytic inverse $C_m^{\langle -1 \rangle}(z)$ of $C_m(z)$ in a n.b. of $z = 0$

i.e., $C_m(C_m^{\langle -1 \rangle}(z)) = z$

$C_m^{\langle -1 \rangle}(C_m(z)) = z$

proof) $C_m(z) = \displaystyle\int \frac{z}{1-zx}\, dm(x) = \sum_{p=0}^{\infty} M_p(m)\, z^{p+1}$, $\forall z$ $|z| \ll 1$

where $M_p(m) := \int_{\mathbb{R}} x^p \, dm(x)$

$\therefore \quad C_m'(0) = M_0(m) = 1 \neq 0$

$\therefore \quad C_m(z)$ is univalent in a n.b. of $z=0$

$\rightsquigarrow \quad \exists \; C_m^{\langle -1 \rangle}(z)$ $\blacksquare$

<u>Rem</u> : $C_m(0) = 0 \quad \rightsquigarrow \quad C_m^{\langle -1 \rangle}(0) = 0$ & $z=0$ is a simple zero

$\therefore \quad \dfrac{1}{C_m^{\langle -1 \rangle}(z)}$ has a simple pole at $z=0$

$$\& \quad \lim_{z \to 0} \frac{z}{C_m^{\langle -1 \rangle}(z)} = \lim_{w \to 0} \frac{C_m(w)}{C_m^{\langle -1 \rangle}(C_m(w))} = \lim_{w \to 0} \frac{C_m(w)}{w} = M_0(m) = 1$$

$$\therefore \quad \frac{1}{C_m^{\langle -1 \rangle}(z)} = \frac{1}{z} + \boxed{\text{analytic function}}$$

<u>Def</u> : The <u>R-transform</u> $R_m(z)$ of m

$$R_m(z) := \frac{1}{C_m^{\langle -1 \rangle}(z)} - \frac{1}{z}$$

<u>Rem</u> : $C_m^{\langle -1 \rangle}(z) = \dfrac{1}{R_m(z) + \frac{1}{z}} \quad \rightsquigarrow \quad G_m\left(R_m(z) + \frac{1}{z} \right) = z$

$$R_m(G_m(z)) + \frac{1}{G_m(z)} = z$$

Def: The quantized R-transform $R_m^{quant}(z)$ of m

$$R_m^{quant}(z) := R_m(z) + \frac{1}{z} - \frac{1}{1-e^{-z}}$$

$$R_m^{quant}(G_m(z)) + \frac{1}{1-e^{-G_m(z)}} = z$$

Rem: $U[0,1] :=$ the uniform measure on $[0,1]$

$$\rightsquigarrow \quad R_{U[0,1]}(z) = -\frac{1}{z} + \frac{1}{1-e^{-z}} \quad \therefore \quad R_m^{quant}(z) = R_m(z) - R_{U[0,1]}(z)$$

$$\because) \quad G_{U[0,1]}(z) = \int_0^1 \frac{1}{z-x}\,dx = \Big[-\log(z-x)\Big]_0^1$$

$$= -\log(z-1) + \log(z)$$

$$= -\log(1-z^{-1})$$

$$\therefore \quad -\log\left(1 - \frac{1}{R_{U[0,1]}(z) + \frac{1}{z}}\right) = z$$

$$1 - \frac{1}{R_{U[0,1]}(z) + \frac{1}{z}} = e^{-z}$$

$$R_{U[0,1]}(z) = \frac{1}{1-e^{-z}} - \frac{1}{z} \quad //$$

Def $m_1, m_2 \in \mathcal{P}(\mathbb{R}) \rightsquigarrow$

- $m_1 \boxplus m_2 \in \mathcal{P}(\mathbb{R})$: free convolution
  if $R_{m_1 \boxplus m_2}(z) = R_{m_1}(z) + R_{m_2}(z)$
- $m_1 \otimes m_2 \in \mathcal{P}(\mathbb{R})$: quantized free convolution
  if $R^{quant}_{m_1 \otimes m_2}(z) = R^{quant}_{m_1}(z) + R^{quant}_{m_2}(z)$

§4 Characters and random probability measures on $\mathbb{R}$

Recall $\mathrm{Ch}(G_N) \cong \mathcal{P}(\widehat{G_N})$, $\quad f = \sum_{\lambda \in \widehat{G_N}} P(\lambda) \frac{1}{d_\lambda} \chi^G_\lambda$

& $\widehat{G_N} \subseteq \widehat{U(N)} \cong \{ \lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \in \mathbb{Z}^N \}$

$\rightsquigarrow$ We will define $m^G, m^G_{PP} : \widehat{G_N} \to \mathcal{P}(\mathbb{R})$

$\rightsquigarrow$ $\forall P \in \mathcal{P}(\widehat{G_N})$, $m^G[P]$, $m^G_{PP}[P]$ $\cdots$ push forward measures on $\mathcal{P}(\mathbb{R})$

Def: $m^G : \widehat{G_N} \to \mathcal{P}(\mathbb{R})$

$\boxed{G=A}$ $\quad m^A[\lambda] := \frac{1}{N} \sum_{j=1}^{N} \delta\left(\frac{\lambda_j + N - j}{N}\right)$

$\boxed{G=B,C,D}$ $\quad m^G[\lambda] := \frac{1}{2N} \sum_{j=1}^{N} \left(\delta_{\left(\frac{\lambda_j + 2N - j}{2N}\right)} + \delta_{\left(\frac{j - \lambda_j}{2N}\right)} \right)$

} Counting measure

↑ motivation は random tiling にあるらしい $\rightsquigarrow$ 状況によって色々と調整する

Rem $\lambda_1 \geq \cdots \geq \lambda_N \rightsquigarrow \frac{\lambda_1 + N - 1}{N} > \cdots > \frac{\lambda_N}{N} \rightsquigarrow \mathrm{Supp}(m^A[\lambda]) \subset \left[\frac{\lambda_N}{N}, \frac{\lambda_1 + N - 1}{N}\right]$

& $\frac{\lambda_1 + 2N - 1}{2N} > \cdots > \frac{\lambda_N + N}{2N}$,

$\frac{N - \lambda_N}{2N} > \cdots > \frac{1 - \lambda_1}{2N}$

$\rightsquigarrow \mathrm{Supp}(m^G[\lambda]) \subset \left[\frac{\lambda_1 + 2N - 1}{2N}, \frac{1 - \lambda_1}{2N}\right]$

Def : $m^G_{PP} : \hat{G}_N \longrightarrow \mathcal{P}(\mathbb{R})$ ... the Perelmov–Popov measure

s.t. $\forall p \geq 1$, $\displaystyle\int_{\mathbb{R}} x^p \, dm^G_{PP}[\lambda](x) = \frac{1}{\hat{N}^{p+1}} C^G_p(\lambda)$

where $\hat{N} = \begin{cases} N & G = A \\ 2N+1 & G = B \\ 2N & G = C, D \end{cases}$

Rem By Thm 2.1, $\forall p \geq 1$,

$$\int_{\mathbb{R}} x^p \, dm^G_{PP}[\lambda](x) = \frac{1}{\hat{N}^{p+1}} \sum_{j \in I_{G_N}} \Lambda_j^p \, \xi_j$$

$$= \frac{1}{\hat{N}} \sum_{j \in I_{G_N}} \left(\frac{\Lambda_j}{\hat{N}}\right)^p \xi_j$$

$$\therefore \quad m^G_{PP}[\lambda] = \frac{1}{\hat{N}} \sum_{j \in I_{G_N}} \delta\left(\frac{\Lambda_j}{\hat{N}}\right) \cdot \xi_j$$

Rem :

$$\hat{C}^G_p = \sum_{i_1, \dots, i_p} X_{i_1 i_2} X_{i_2 i_3} \cdots X_{i_p i_1}$$

$$= (\mathrm{Tr} \otimes \mathrm{id})(\hat{X}^p), \quad \text{where} \quad \hat{X} = \sum_{i,j \in I_{G_N}} e_{ij} \otimes X_{ij}$$

& $C^G_p(\lambda) = \mathrm{tr}_{V_\lambda}\left(\pi_\lambda(\hat{C}^G_p)\right) = (\mathrm{Tr} \otimes \mathrm{tr}_{V_\lambda} \circ \pi_\lambda)(\hat{X}^p)$, where $\mathrm{tr}_{V_\lambda} = \frac{1}{d_\lambda} \mathrm{Tr}_{V_\lambda}$

$\rightarrow \quad \frac{1}{N^{p+1}} C_p^G(\lambda) = (\mathrm{tr} \otimes \mathrm{tr}_{V_\lambda} \circ \pi_\lambda)\left(\left(\frac{1}{\hat{N}} \tilde{X}\right)^p \right)$

$\rightsquigarrow \quad m^G_{pp}[\lambda]$ is the spectral measure of $\frac{1}{\hat{N}} \tilde{X}$ w.r.t. $\mathrm{tr} \otimes \mathrm{tr}_{V_\lambda} \circ \pi_\lambda$

- Explicit formula of $m^G_{pp}[\lambda]$

$\forall \lambda \in \hat{G}_N, \qquad \lambda^{(i\pm)} := \lambda \pm \delta_i$

$$\rightsquigarrow \; d_{\lambda^{(i\pm)}} := \begin{cases} d_{\lambda^{(i\pm)}} & \text{if } \lambda^{(i\pm)} \in \hat{G}_N \\ 0 & \text{otherwise} \end{cases}$$

<u>Lem</u> $\quad \forall \lambda \in \hat{G}_N,$

By the Weyl dimension formula

$$\left(\boxed{G=A} \quad \xi_i = \prod_{j=1}^{N} \left(1 - \frac{1}{\lambda_i - i - \lambda_j + j}\right) = \prod_{j=1}^{N} \frac{\lambda_i - i - 1 - \lambda_j + j}{\lambda_i - i - \lambda_j + j} \right)$$

- $$m^A_{pp}[\lambda] = \frac{1}{N} \sum_{i=1}^{N} \frac{d_{\lambda^{(i-)}}}{d_\lambda} \delta\left(\frac{\lambda_i + N - i}{N}\right)$$

- $G \neq A$,
$$m^G_{pp}[\lambda] = \frac{1}{\hat{N}} \sum_{i=1}^{N} \left\{ \frac{d_{\lambda^{(i-)}}}{d_\lambda} \delta\left(\frac{\lambda_i + 2N - i + \varepsilon_G - 1}{\hat{N}}\right) + \frac{d_{\lambda^{(i+)}}}{d_\lambda} \delta\left(\frac{i - \lambda_i - 1}{\hat{N}}\right) \left(+ \underbrace{\delta\left(\frac{N}{\hat{N}}\right)}_{\text{only type B}} \right) \right\}$$

where $\varepsilon^G = \begin{cases} 1 & G=B \\ 2 & G=C \\ 0 & G=D \end{cases}$

Def: $P \in \mathcal{P}(\widehat{G}_N)$,

$$S_P^{G_N}(z_1, \dots, z_N) := \sum_{\lambda \in \widehat{G}_N} P(\lambda) \frac{\chi_\lambda^G(z_1, \dots, z_N)}{\chi_\lambda^G(1^N)}$$

$(1^N) = (1, \dots, 1)$

$S_P^{G_N}$: the character generating function of P

Rem: LHS converges uniformly on $\mathbb{T}^N$

$\rightsquigarrow$ In what follows, we always assume that the LHS converges uniformly in a open neighborhood of $(1^N) \in \mathbb{C}^N$ & $S_P^{G_N}(z_1, \dots, z_N)$ is analytic in this domain (e.g. Supp(P) is finite)

Q How to extract the information of $m^G[P]$ from $S_P^{G_N}(z)$?

Def: $V^G(z_1, \dots, z_N) :=$ the denominator of the Weyl character formula of $\chi_\lambda^G(z_1, \dots, z_N)$

e.g. $$V^A(z_1, \dots, z_N) = \det\left[z_i^{N-j}\right]_{i,j=1}^N = \prod_{1 \le i < j \le N} (z_i - z_j)$$

$\rightsquigarrow \quad \forall\, k \geq 1,$

$$\mathcal{D}_k^{G_N} := \frac{1}{V^G(z_1,\dots,z_N)} \sum_{i=1}^{N} \left(z_i \frac{\partial}{\partial z_i} \right)^k \cdot V^G(z_1,\dots,z_N)$$

Euler operator E_{z_i}

Lem 4.1 $\forall\, k \geq 1$,

$\boxed{G=A}$

$$\mathcal{D}_k^{U(N)} \chi_\lambda^A(z_1,\dots,z_N) = \sum_{j=1}^{N} l_j^k \, \chi_\lambda^A(z_1,\dots,z_N)$$

$\boxed{G\neq A}$

$$\mathcal{D}_{2k}^{G_N} \chi_\lambda^G(z_1,\dots,z_N) = \sum_{j=1}^{N} l_j^k \, \chi_\lambda^{G_N}(z_1,\dots,z_N),$$

where $l_j := \lambda_j + N - j + \varepsilon^G/2$, $\quad \varepsilon^G = \begin{cases} 0 & G = A, D \\ 1 & G = B \\ 2 & G = C \end{cases}$

Proof)

$\boxed{G=A}$ Recall : $\chi_\lambda^A(z) = \dfrac{\det\left[z_i^{l_j} \right]_{i,j=1}^N}{V^A(z)}$

$$\rightsquigarrow \mathcal{D}_k^{U(N)} \chi_\lambda^A(z) = \frac{1}{V^A(z)} \sum_{i=1}^{N} E_{z_i}^k \cdot \det\left[z_i^{l_j} \right]_{i,j=1}^N$$

$$= \frac{1}{V^A(z)} \sum_{i=1}^{N} \det \begin{bmatrix} \cdots z_1^{l_j} \cdots \\ \cdots l_j^k z_i^{l_j} \cdots \\ \cdots z_N^{l_j} \cdots \end{bmatrix}$$

$$= \frac{1}{V^A(z)} \sum_{i=1}^{N} \sum_{j=1}^{N} l_j^k \, z_i^{l_j} \, C_{ij}$$

C_{ij}: the adjugate matrix of $[z_i^{l_j}]_{ij}$

$$= \frac{1}{V^A(z)} \sum_{j=1}^{N} l_j^k \cdot \sum_{i=1}^{N} z_i^{l_j} C_{ij}$$

$$= \left(\sum_{j=1}^{N} l_j^k \right) \cdot \chi_\lambda^A(z)$$

$\boxed{G \neq A}$ $\because$ $\chi_\lambda^B(z) = \dfrac{\det\left[z_i^{l_j} - z_i^{-l_j} \right]_{i,j=1}^{N}}{V^B(z)}$

$$\rightsquigarrow \quad E_{z_i}^{2k} \left(z_i^{l_j} - z_i^{-l_j} \right) = l_j^{2k} z_i^{l_j} - (-l_j)^{2k} \cdot z_i^{-l_j}$$

$$= l_j^{2k} \left(z_i^{l_j} - z_i^{-l_j} \right)$$

$\square$

4.X

Prop: $\forall\, p \in \mathcal{P}(\widehat{G_N})$ s.t. $S_p^{G_N}$ is well-defined, $\forall\, k, m \geq 1$,

$\boxed{G = A}$

$$\mathbb{E}\left[M_k (m^A[p])^m \right] = \frac{1}{N^{m(k+1)}} \left(\mathcal{D}_k^{U(N)} \right)^m S_p^{U(N)}(z) \Big|_{z = (1^N)}$$

M_k: the k-th moment

$\boxed{G \neq A}$

$$\hat{m}^G[\lambda] := \frac{1}{2N} \sum_{i=1}^{N} \left\{ \delta\left(\frac{l_i}{2N}\right) + \delta\left(-\frac{l_i}{2N}\right) \right\} = m^G[\lambda]\left(\cdot - \frac{N + \varepsilon_G/2}{2N} \right)$$

$$\mathbb{E}\left[M_{2k} (\hat{m}^G[p])^m \right] = \frac{1}{2^{2mk} N^{m(2k+1)}} \left(\mathcal{D}_{2k}^{G_N} \right)^m S_p^{G_N}(z) \Big|_{z = (1^N)}$$

Proof of Type A)

By Lem 4.1, $\frac{1}{N^{k+1}} \mathcal{D}_k^{U(N)} \chi_\lambda^A(z) = M_k(m^A[p]) \chi_\lambda^A(z)$

$\therefore \mathbb{E}[M_k(m^A[p])^m] = \sum_\lambda M_k(m^A[p])^m P(\lambda)$

$$= \sum_\lambda P(\lambda) \frac{1}{N^{m(k+1)}} (\mathcal{D}_k^{U(N)})^m \frac{\chi_\lambda^A(z)}{\chi_\lambda^A(1^N)} \Big|_{z=(1^N)}$$

$$= \frac{1}{N^{m(k+1)}} (\mathcal{D}_k^{U(N)})^m S_p^{U(N)}(z) \Big|_{z=(1^N)} \qquad \boxtimes$$

Q How to extract the information of $m_{pp}^G[\lambda]$ from $S_p^{G_N}(z)$?

Def: $\forall k \geq 1$,

$$\mathcal{D}_k^{PP,U(N)} := \frac{1}{V^A(z)} \sum_{i=1}^N \frac{\partial}{\partial z_i} E_{z_i}^{k-1} \cdot V^A(z)$$

$$\left(z_i \frac{\partial}{\partial z_i} \right)$$

$G \neq A$

$$\rightsquigarrow \mathcal{D}_k^{PP,G_N} := \frac{1}{V^G(z)} \sum_{i=1}^N (z_i^{-1} \pm z_i) E_{z_i}^k \cdot V^G(z)$$

$$\begin{cases} + & k: \text{even} \\ - & k: \text{odd} \end{cases}$$

Lem 4.2

$\boxed{G=A}$

$$\mathcal{D}_k^{PP,\,U(N)} \chi^A_\lambda = \sum_{j=1}^{N} l_j^k \, \chi^A_{\lambda(j-)}$$

$\boxed{G\neq A}$

$$\mathcal{D}_k^{PP,\,G_N} \chi^G_\lambda = \sum_{j=1}^{N} l_j^k \, \chi^G_{\lambda(j-)} + (-l_j)^k \, \chi^G_{\lambda(j+)}$$

Proof)

$\boxed{G=A}$

$$\mathcal{D}_k^{PP,\,U(N)} \chi^A_\lambda(z)$$

$$= \frac{1}{V^A(z)} \sum_{i=1}^{N} \frac{\partial}{\partial z_i} \cdot E_{z_i}^{k-1} \cdot \det\left[z_i^{l_j} \right]_{i,j=1}^{N}$$

$$= \frac{1}{V^A(z)} \sum_{i=1}^{N} \det\left[\begin{array}{c} \text{---} \\ \cdots\ l_j^k \cdot z_i^{l_j-1} \cdots \\ \text{---} \end{array} \right]$$

$$= \frac{1}{V^A(z)} \sum_{i=1}^{N} \sum_{j=1}^{N} l_j^k \cdot z_i^{l_j-1} \cdot C_{ij}$$

the adjugate matrix of $\left[z_i^{l_j} \right]_{i,j=1}^{N}$

$$= \frac{1}{V^A(z)} \sum_{j=1}^{N} l_j^k \sum_{i=1}^{N} z_i^{l_j-1} C_{ij}$$

$$= \frac{1}{V^A(z)} \sum_{j=1}^{N} l_j^k \det\left[\begin{array}{ccc} | & \vdots & | \\ | & z_i^{l_j-1} & | \\ | & \vdots & | \end{array} \right]$$

$$= \sum_{j=1}^{N} l_j^k \cdot \chi^A_{\lambda(j-)}(z)$$

$\boxed{G \neq A}$

$$(z_i^{-1} \pm z_i)\, E_{z_i}^{k}\, (z_i^{l_j} \pm z_i^{-l_j})$$

$$= (z_i^{-1} \pm z_i)\left(l_j^{k} z_i^{l_j} \pm (-l_j)^{k} z_i^{-l_j} \right)$$

$$= l_j^{k} \left(z_i^{l_j - 1} \pm z_i^{-(l_j-1)} \right) + (-l_j)^{k} \left(z_i^{l_j+1} \pm z_i^{-(l_j+1)} \right)$$

$\rightsquigarrow$ the same computation ///

<u>Prop</u> $\forall\, p \in \mathcal{P}(\hat{G}_N)$, $\forall\, k \geq 1$,

$\boxed{G = A}$ $$\mathbb{E}\left[M_k(m^{A}_{pp}[\lambda]) \right] = \frac{1}{N^{k+1}} \mathcal{D}_k^{pp,\,U(N)} S_p^{U(N)}(z) \Big|_{z=(1^N)}$$

$\boxed{G \neq A}$ $$\mathbb{E}\left[M_k(\hat{m}^{G}_{pp}[\lambda]) \right] = \frac{1}{\hat{N}^{k+1}} \mathcal{D}_k^{pp,\,G_N} S_p^{G_N}(z) \Big|_{z=(1^N)}$$

where $$\hat{m}^{G}_{pp}[\lambda] := \begin{cases} m^{B}_{pp}[\lambda]\left(\cdot + \frac{N-1/2}{2N+1}\right) - \frac{1}{2N+1}\delta\left(\frac{1/2}{2N+1}\right) \\ m^{C}_{pp}[\lambda]\left(\cdot + \frac{N}{2N}\right) \\ m^{D}_{pp}[\lambda]\left(\cdot + \underbrace{\frac{N-1}{2N}}\right) \end{cases}$$

$$l_j = \lambda_j + $$
$$-l_j = \lambda_{-j} + $$

Proof in type A)

By Lem 4.2,

$$\frac{1}{N^{k+1}} \mathcal{D}_k^{PP, U(N)} \frac{\chi_\lambda^A(z)}{\chi_\lambda^A(1^N)} \Big|_{z=(1^N)}$$

$$= \frac{1}{N} \sum_{j=1}^{N} \left(\frac{\lambda_j + N - j}{N} \right)^k \cdot \frac{\chi^A_{\lambda(j-)}(z)}{\chi^A_\lambda(1^N)} \Big|_{z=(1^N)}$$

$$= \frac{1}{N} \sum_{j=1}^{N} \left(\frac{\lambda_j + N - j}{N} \right)^k \frac{d_{\lambda(j-)}}{d_\lambda} = M_k(m^A_{PP}[\lambda])$$

$$\rightsquigarrow \mathbb{E}[M_k(m^A_{PP}[P])]$$

$$= \sum_\lambda P(\lambda) M_k(m^A_{PP}[\lambda])$$

$$= \sum_\lambda P(\lambda) \cdot \frac{1}{N^{k+1}} \mathcal{D}_k^{PP, U(N)} \frac{\chi^A_\lambda(z)}{\chi^A_\lambda(1^N)} \Big|_{z=(1^N)}$$

$$= \frac{1}{N^{k+1}} \mathcal{D}_k^{PP, U(N)} S_P^{U(N)}(z) \Big|_{z=(1^N)} \qquad \blacksquare$$

@ Relation between m^G & m^G_{PP}

Recall $\quad C_\lambda^G(z) := \sum_{p=0}^{\infty} c_p^G(\lambda) z^p = \left(1 + \frac{\beta_G z}{z - (2N + \varepsilon_G - 1) z}\right) \frac{1 - \Pi_\lambda^G(z)}{z}$

where $\quad \Pi_\lambda^G(z) := \prod_{i \in I_{G_N}} \left(1 - \frac{z}{1 - \Delta_i z}\right)$

Lem 4.3 $z \in \mathbb{C} \setminus \mathbb{R}$: $|z| \ll 1$

$$\frac{1}{\hat{N}} \log \Pi^G_\lambda\left(\frac{z}{\hat{N}}\right) = \int_{\mathbb{R}} \log\left(1 - \frac{z}{\hat{N}} \cdot \frac{1}{1 - xz}\right) d\tilde{m}^G[\lambda](x)$$

where
$$\tilde{m}^G[\lambda] = \begin{cases} m^A[\lambda] & G = A \\ \frac{1}{\hat{N}} \sum_{i=1}^{N} \left\{ \delta\left(\frac{\lambda_i + 2N - i + \varepsilon_G - 1}{\hat{N}}\right) + \delta\left(\frac{i - \lambda_i - 1}{\hat{N}}\right) \right. & G \neq A \\ \qquad \left. \left(+ \frac{1}{2N+1} \delta\left(\frac{N}{2N+1}\right)\right) \right\} & \end{cases}$$

$\uparrow$ only type B

Rem 4.4 $\lambda(N) \in \hat{G}_N$ $(N \geq 1)$

$m^G[\lambda(N)] \to m$ weakly as $N \to \infty$

$\iff$ $\tilde{m}^G[\lambda(N)] \to m$ weakly as $N \to \infty$

Proof (in type A)

$$\Pi^A_\lambda\left(\frac{z}{N}\right) = \prod_{i=1}^{N} \left(1 - \frac{z}{N} \cdot \frac{1}{1 - \frac{\lambda_i + N - i}{N} \cdot z}\right)$$

$$\therefore \frac{1}{N} \log \Pi^A_\lambda\left(\frac{z}{N}\right) = \frac{1}{N} \sum_{i=1}^{N} \log\left(1 - \frac{z}{N} \frac{1}{1 - \frac{\lambda_i + N - i}{N} \cdot z}\right)$$

$$= \int_{\mathbb{R}} \log\left(1 - \frac{z}{N} \frac{1}{1 - zx}\right) dm^A[\lambda](x)$$

$\square$

$\underline{\text{Thm } 4.5}$ $P_N \in \mathcal{P}(\widehat{G}_N)$ $(N \geq 1)$,

in the moments sense

Assume $m^G[P_N] \to m$ weakly, in probability as $N \to \infty$

(i.e., $\forall f \in C_b(\mathbb{R})$, $\int_{\mathbb{R}} f(x)\, dm^G[P_N](x) \to \int_{\mathbb{R}} f(x)\, dm(x)$ in probability)

Then $m^G_{pp}[P_N] \to Q(m)$ in the moment sense, in probability as $N \to \infty$

& $Q(m)$ is given by

$$1 - G_{Q(m)}(z) = \exp(-G_m(z))$$

$\underline{\text{Rem}}$ ∘ $m \mapsto Q(m)$ is a modification of the Markov-Krein Correspondence.

But, this is not surjective.

∘ Q is defined on $\{ m \in \mathcal{P}(\mathbb{R}) \mid \text{compact supp},\ m \ll dx,\ \frac{dm}{dx} \leq 1 \text{ a.e.} \}$

"idea of proof"

By Rem 4.4, $\widetilde{m}^G[P_N] \to m$ weakly, in probability

$\rightsquigarrow$ $\underline{\text{Recall}}$ $\lambda(N) \sim P_N$

$$\Rightarrow \quad G_{m^G_{pp}[\lambda(N)]}(z^{-1}) = \sum_{p=0}^{\infty} M_p(m^G_{pp}[\lambda(N)])\, z^{p+1}$$

$$= \frac{z}{\hat{N}} C^{G}_{\lambda(N)} \left(\frac{z}{\hat{N}}\right)$$

$$= \left(1 + \frac{\beta_G z/\hat{N}}{2-(2N+\varepsilon_G-1) z/\hat{N}}\right) \left(1 - \Pi^{G}_{\lambda(N)}\left(\frac{z}{\hat{N}}\right)\right)$$

By Lem 4.3,

$$\Pi^{G}_{\lambda(N)}\left(\frac{z}{\hat{N}}\right) = \exp\left(\hat{N} \int_{\mathbb{R}} \log\left(1 - \frac{z}{\hat{N}} \frac{1}{1-zx}\right) d\tilde{m}^{G}[\lambda(N)](x)\right)$$

$$= \exp\left(- \int_{\mathbb{R}} \left(\frac{z}{1-zx} + O\left(\frac{1}{\hat{N}}\right)\right) d\tilde{m}^{G}[\lambda(N)](x)\right)$$

$$\longrightarrow \exp\left(- \int_{\mathbb{R}} \frac{z}{1-zx} \, dm(x)\right)$$

$$\therefore \quad G_{m^{G}_{pp}[\lambda(N)]}(z^{-1}) \longrightarrow 1 - \exp\left(-G_m(z^{-1})\right)$$

Rem : $iy\left(1 - \exp\left(-G_m(iy)\right)\right)$

$$= iy\, G_m(iy) + o(y) \longrightarrow 1 \quad \text{as } y \to \infty$$

//

Thm $\forall m_1, m_2 \in \mathcal{P}(\mathbb{R})$ s.t. $Q(m_1), Q(m_2)$ are well-defined

Then, $\exists\ m_1 \otimes m_2$, $Q(m_1 \otimes m_2)$ is well-defined

& $Q(m_1 \otimes m_2) = Q(m_1) \boxplus Q(m_2)$

proof) existence of $m_1 \otimes m_2$ $\leadsto$ later (using approximation)

Recall : $R_{Q(m)}(G_{Q(m)}(z)) = z - \dfrac{1}{G_{Q(m)}(z)}$

$$= z - \frac{1}{1-\exp(-G_m(z))} = R_m^{quant}(G_m(z))$$

$$\therefore \quad R_{Q(m)}(1-\exp(-G_m(z))) = R_m^{quant}(G_m(z))$$

$$\leadsto \quad R_{Q(m_1 \otimes m_2)}(u) = R^{quant}_{m_1 \otimes m_2}(w) \qquad u = 1 - e^{-w}$$

$$= R^{quant}_{m_1}(w) + R^{quant}_{m_2}(w)$$

$$= R_{Q(m_1)}(u) + R_{Q(m_2)}(u) \qquad \square$$

§5 Asymptotic representation theory (original asymptotic regime)

- Vershik–Kerov's ergodic method:

$$G_\infty := \varinjlim G_N \quad , \text{ where } G = A, B, C \ (D)$$

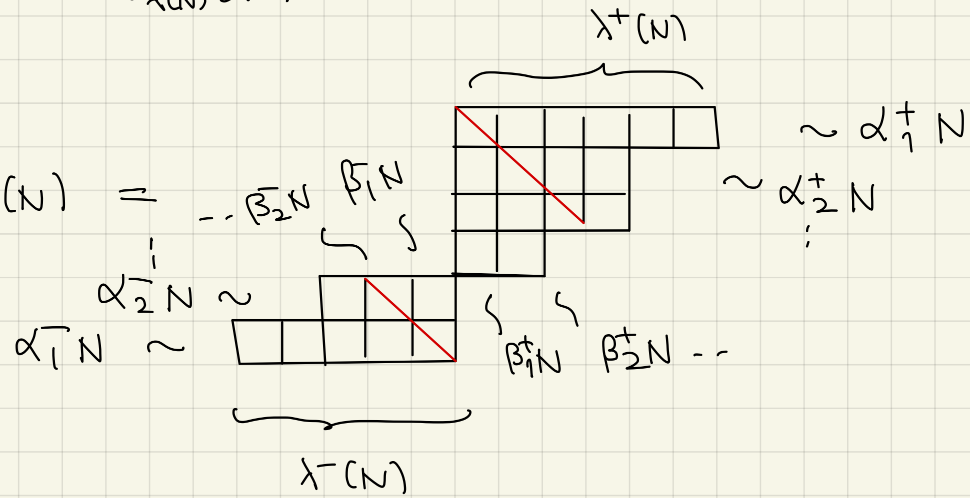
$\forall \chi \in \mathrm{ex\,Ch}(G_\infty)$, $\exists\ \lambda(N) \in \widehat{G_N}$ $(N \geq 1)$ s.t.

$$\forall k \geq 1, \quad \chi|_{G_k} = \lim_{N\to\infty} \frac{\chi^{G_N}_{\lambda(N)}|_{G_k}}{\chi^{G_N}_{\lambda(N)}(1^N)} \quad \text{uniformly}$$

<u>Thm</u> (Okounkov–Olshanski '98) : $\boxed{G = A}$

$\lambda(N) \in \widehat{U(N)}$ $(N \geq 1)$, TFAE

- $\exists \lim\limits_{N\to\infty} \dfrac{\chi^A_{\lambda(N)}|_{U(k)}}{\chi^A_{\lambda(N)}(1^N)}$ uniformly $\forall k \geq 1$

- $\lambda(N) =$ 

& $|\lambda^\pm(N)| \sim \delta^\pm N$

Then,
$$\lim_{N\to\infty} \frac{\chi^A_{\lambda(N)}(z_1, \ldots, z_k, 1^{N-k})}{\chi^A_{\lambda(N)}(1^N)} = \Phi^A_\omega(z_1) \cdots \Phi^A_\omega(z_k),$$

$$\Phi^A_w(z) := e^{\gamma^+(z-1)+\gamma^-(z^{-1}-1)} \prod_{i=1}^{\infty} \frac{1+\beta_i^+(z-1)}{1-\alpha_i^+(z-1)} \frac{1+\beta_i^-(z^{-1}-1)}{1-\alpha_i^-(z^{-1}-1)}$$

$$\left(\begin{array}{l} w = (\alpha^+, \beta^+, \alpha^-, \beta^-, \delta^+, \delta^-), \\ \gamma^\pm := \delta^\pm - \sum_{i=1}^{\infty} (\alpha_i^\pm + \beta_i^\pm) \end{array} \right)$$

Thm (Edrei–Voiculescu theorem (Voiculescu, Vershik–Kerov, Boyer, ---))

$\mathrm{ex}\,\mathrm{Ch}(U(\infty))$

$$\cong \Omega^A := \left\{ w = (\alpha^+, \beta^+, \alpha^-, \beta^-, \gamma^+, \gamma^-) \in (\mathbb{R}^\infty_{\geq 0})^4 \times \mathbb{R}^2_{\geq 0} \;\middle|\; \begin{array}{l} \alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots) \\ \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots) \\ \sum_i \alpha_i^\pm + \beta_i^\pm < \infty, \\ \beta_i^+ + \beta_i^- \leq 1 \end{array} \right\}$$

Moreover, $\forall w \in \Omega^A \longleftrightarrow \chi^A_w \in \mathrm{ex}\,\mathrm{Ch}(U(\infty))$,

$$\chi^A_w\left(\begin{pmatrix} z_1 & & \\ & z_2 & \\ & & \ddots \end{pmatrix}\right) = \prod_{i=1}^{\infty} \Phi^A_w(z_i)$$

Thm (type BC version (Okounkov–Olshanski '06))

$\mathrm{ex}\,\mathrm{Ch}(G_\infty) \quad (G = B \text{ or } C)$

$$\cong \Omega^{BC} := \left\{ w = (\alpha, \beta, \gamma) \in (\mathbb{R}^\infty_{\geq 0})^2 \times \mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{l} \alpha = (\alpha_1 \geq \alpha_2 \geq \cdots) \\ \beta = (\beta_1 \geq \beta_2 \geq \cdots) \\ \sum_i \alpha_i + \beta_i < \infty \end{array} \right\}$$

Moreover, $\forall\, w \in \Omega^{BC} \longleftrightarrow \chi_w^G \in \mathrm{ex}\,\mathrm{Ch}(G_\infty)$,

$$\chi_w^G\left(\begin{pmatrix} D(z_1) & & \\ & D(z_2) & \\ & & \ddots \end{pmatrix}\right) = \prod_{i=1}^{\infty} \Phi_w^{BC}(z_i),$$

where $\Phi_w^{BC}(z) := e^{\gamma(z-1)} \prod_{i=1}^{\infty} \dfrac{1+\frac{\beta_i(2-\beta_i)}{2}(x-1)}{1+\frac{\alpha_i(2-\alpha_i)}{2}(x-1)}$

Rem $G = B, C$, $\forall\, \lambda \in \widehat{G}_N$

$$\chi_\lambda^G(z_1,\dots,z_N) \propto \mathcal{P}_\lambda\left(\frac{z_1+z_1^{-1}}{2},\dots,\frac{z_N+z_N^{-1}}{2}; a, b\right),$$

where $\mathcal{P}_\lambda(x_1,\dots,x_N; a,b) = \dfrac{\det\left[P_{\lambda_i+N-i}(x_j; a,b)\right]_{i,j=1}^N}{\det\left[x_j^{N-i}\right]_{i,j=1}^N}$,

& $P_n(x; a,b) :=$ the Jacobi polynomial of degree n

& $(a,b) = \begin{cases} \left(\frac{1}{2}, -\frac{1}{2}\right) & G = B \\ \left(\frac{1}{2}, \frac{1}{2}\right) & G = C \end{cases}$

@ Other asymptotic regimes

- Asymptotic representation theory : $\lambda_i(N) = O(N)$ as $N \to \infty$

  & Bufetov–Gorin

- Biane '95 : $\frac{1}{N}\sum_{i=1}^{N} \delta_{\varepsilon_N \cdot \lambda_i(N)}$ where $\varepsilon_N = o(N^{-\infty})$ i.e., $\varepsilon_N N^a \to 0$ $\forall a \geq 1$

- Collins – Śniady '09 : $\frac{1}{N}\sum_{i=1}^{N}\delta_{\varepsilon_N(\lambda_i(N)+N-i)}$,

  where $\varepsilon_N = o(\frac{1}{N})$

- Borodin – Bufetov – Olshanski '15

  $w(N) = (\alpha^+(N), \beta^+(N), \alpha^-(N), \beta^-(N), \gamma^+(N), \gamma^-(N)) \in \Omega^A \quad (N \geq 1)$

  s.t.
  - $\exists \lim_{N\to\infty} \frac{\gamma^\pm(N)}{N} = \gamma^\pm$
  - $\exists$ finite measures $\mathfrak{A}^\pm, \mathfrak{B}^\pm$ on $\mathbb{R}$ with compact supports

    s.t. $\mathfrak{A}_N^\pm := \frac{1}{N}\sum_{i=1}^{\infty}\delta_{\alpha_i^\pm(N)} \to \mathfrak{A}^\pm$

    $\mathfrak{B}_N^\pm := \frac{1}{N}\sum_{i=1}^{\infty}\delta_{\beta_i^\pm(N)} \to \mathfrak{B}^\pm$ weakly

    only nonzero term
  - $\exists\ C_1, C_2 > 0$ s.t.

    $\mathrm{supp}(\mathfrak{A}_N^\pm) \subset [0, C_1]$, $\mathrm{supp}(\mathfrak{B}_N^\pm) \subset [0, C_1]$

    $|\mathrm{supp}(\mathfrak{A}_N^\pm)| < C_2 N$, $|\mathrm{supp}(\mathfrak{B}_N^\pm)| < C_2 N$

<u>Rem</u> $\lambda \in \widehat{U(N)} \to$ "modified" Frobenius coordinate of $\lambda(N) \in \Omega^A$

<u>Prop 5.1</u> : Under the above assumption,

$$\lim_{N\to\infty}\frac{1}{N}\log\Phi^A_{w(N)}(1+z)$$

$$= \gamma^+ z - \gamma^- \frac{z}{1+z} + \int_{\mathbb{R}} \log(1+zx)\, d\mathcal{B}^+(x) + \int_{\mathbb{R}} \log\left(1 - \frac{zx}{1+z}\right) d\mathcal{B}^-(x)$$

$$- \int_{\mathbb{R}} \log(1-zx)\, d\mathcal{A}^+(x) - \int_{\mathbb{R}} \log\left(1 + \frac{zx}{1+z}\right) d\mathcal{A}^-(x) \quad — \star$$

uniformly on a neighborhood of $z=0$

Proof) Recall : $\Phi^A_W(1+z) = e^{\gamma^+ z - \gamma^- \frac{z}{1+z}} \prod_{i=1}^{\infty} \frac{1+\beta_i^+ z}{1-\alpha_i^+ z} \cdot \frac{1-\beta_i^- \frac{z}{1+z}}{1+\alpha_i^- \frac{z}{1+z}}$

$$\frac{1}{N} \log \Phi^A_{W(N)}(1+z) = \frac{\gamma^+(N)}{N} z - \frac{\gamma^-(N)}{N} \frac{z}{1+z}$$

$$= \frac{\gamma^+(N)}{N} \qquad + \frac{1}{N}\sum_{i=1}^{\infty} \log(1+\beta_i^+(N) z) + \frac{1}{N}\sum_{i=1}^{\infty} \log\left(1-\beta_i^-(N)\frac{z}{1+z}\right)$$

$$+ \frac{1}{N}\sum_{i=1}^{\infty} \log(1-\alpha_i^+(N) z) + \frac{1}{N}\sum_{i=1}^{\infty} \log\left(1+\alpha_i^-(N)\frac{z}{1+z}\right)$$

$$= \frac{\gamma^+(N)}{N} z - \frac{\gamma^-(N)}{N} \frac{z}{1+z}$$

$$+ \int_{\mathbb{R}} \log(1+zx)\, d\mathcal{B}_N^+(x) + \int_{\mathbb{R}} \log\left(1 - \frac{z}{1+z}x\right) d\mathcal{B}_N^+(x)$$

$$+ \int_{\mathbb{R}} \log(1-zx)\, d\mathcal{A}_N^+(x) + \int_{\mathbb{R}} \log\left(1 + \frac{z}{1+z}x\right) d\mathcal{A}_N^-(x)$$

$\to \star$ as $N \to \infty$

$\because$) the integrands are bounded on $[0, |z| C_2]$ $\quad |z| \ll 1$

▨

Thm 5.2 (Bufetov – Borodin – Olshanski '15)

$w(N) \in \Omega^A$ s.t. $\lim_{N\to\infty} \frac{1}{N} \log \Phi^A_{w(N)}(1+z) = P(z)$ uniformly on a n.b. of $z=0$

& $P_N \in \mathcal{P}(\widehat{U(N)})$ associated with $\chi^A_{w(N)}|_{U(N)} \in Ch(U(N))$

$\Rightarrow$ $m^A[P_N] \to {}^\exists \sigma \in \mathcal{P}(\mathbb{R})$ weakly, in probability

s.t. $R^{\text{quant}}_\sigma(z) = e^z P'(e^z - 1)$

Rem: By Thm 4.5, $m^A_{pp}[P_N] \to Q(\sigma)$ # & $R_{Q(\sigma)}(u) = \frac{1}{1-u} P'\left(\frac{-u}{1-u}\right)$

($\therefore$ $R_{Q(\sigma)}(1-e^{w}) = R^{\text{quant}}_\sigma(w)$)

Cor Under the assumption in Prop. 5.1, Thm 5.2,

$$R^{\text{quant}}_\sigma(z) = \gamma^+ e^z - \gamma^- e^{-z} + \int_{\mathbb{R}} \frac{e^z x}{1+(e^z-1)x}\, d\mathcal{B}^+(x) + \int_{\mathbb{R}} \frac{e^{-z} x}{1-(e^{-z}-1)x}\, d\mathcal{B}^-(x)$$

$$- \int_{\mathbb{R}} \frac{e^z x}{1-(e^z-1)x}\, d\Omega^+(x) - \int_{\mathbb{R}} \frac{e^{-z} x}{1+(e^{-z}-1)x}\, d\Omega^-(x)$$

In particular, σ is infinitely divisible in the quantized free sense

(i.e., $\forall n \geq 1$, $\sigma = \sigma_n^{\otimes n}$,

where $\sigma_n \longleftrightarrow \left(\frac{1}{n}\Omega^+, \frac{1}{n}\mathcal{B}^+, \frac{1}{n}\Omega^-, \frac{1}{n}\mathcal{B}^-, \frac{\gamma^+}{n}, \frac{\gamma^-}{n}\right)$)

Rem: $Q(\sigma)$ is infinitely divisible in the free sense

i.e., $\forall n \geq 1$, $Q(\sigma) = Q(\sigma_n)^{\boxplus n}$

- Gorin–Panova '15, Bufetov–Gorin '15

Def $\lambda(N) \in \widehat{G}_N$ $(N \geq 1)$

$(\lambda(N))_{N\geq 1}$: regular if

$\exists\, f : [0,1] \to \mathbb{R}$: piecewise continuous, (non-increasing)

$\exists\, C > 0$

s.t. $\displaystyle\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \left|\frac{\lambda_j(N)}{N} - f\left(\frac{j}{N}\right)\right| = 0$, $\displaystyle\sup_{N\geq 1}\max_j \left|\frac{\lambda_j(N)}{N} - f\left(\frac{j}{N}\right)\right| < C$

Rem $\mathrm{supp}(m^A[\lambda(N)]) \subseteq \left[\frac{\lambda_N(N)}{N}, \frac{\lambda_1(N)+N-1}{N}\right] \subseteq [f(1)-C, \|f\|_\infty + C + 1]$

$\mathrm{Supp}(m^G[\lambda(N)]) \subseteq \left[\frac{1-\lambda_1(N)}{2N}, \frac{\lambda_1(N)+2N-1}{2N}\right] \subseteq \left[-\frac{\|f\|_\infty + C}{2}, \frac{\|f\|_\infty + C}{2} + 1\right]$

Lem 5.3 $(\lambda(N))_{N\geq 1}$: regular

$\Rightarrow$ $m^G[\lambda(N)] \to \exists\, m$ weakly s.t. $\hat{f}(x) = f(x) + 1 - x$ ✓

$$G_m(z) = \int_0^1 \frac{1}{z - \hat{f}(x)}\,dx, \quad G = A$$

$$\left| \frac{1}{2} \int_0^1 \left(\frac{1}{z - \frac{\hat{f}(x)+1}{2}} + \frac{1}{z + \frac{\hat{f}(x)-1}{2}} \right) dx \right.$$

proof in type A) $\because$ $iy \int_0^1 \frac{1}{iy - \hat{f}(x)} dx \to 1$ as $y \to \infty$ $\qquad G \neq A$

$\rightsquigarrow$ It suffices to show : $\lim_{N \to \infty} G_{m^A[\lambda(N)]}(z) = G_m(z)$ pt-wise

$$G_{m^A[\lambda(N)]}(z)$$

$$= \frac{1}{N} \sum_{j=1}^{N} \frac{1}{z - \frac{\lambda_j(N)+N-j}{N}}$$

$$= \frac{1}{N} \sum_{j=1}^{N} \frac{1}{z - \hat{f}(\frac{j}{N})} + \frac{1}{N} \sum_{j=1}^{N} \left\{ \frac{1}{z - \frac{\lambda_j(N)+N-j}{N}} - \frac{1}{z - \hat{f}(\frac{j}{N})} \right\}$$

$\because$

$$|\ast| \leq \frac{1}{N} \sum_{j=1}^{N} \left| \frac{\hat{f}(\frac{j}{N}) - \frac{\lambda_j(N)+N-j}{N}}{\left(z - \frac{\lambda_j(N)+N-j}{N}\right) \cdot \left(z - \hat{f}(\frac{j}{N})\right)} \right|$$

$$\leq \frac{1}{|\mathrm{Im}(z)|^2} \frac{1}{N} \sum_{j=1}^{N} \left| f(\tfrac{j}{N}) - \frac{\lambda_j(N)}{N} \right| \longrightarrow 0 \quad \text{as } N \to \infty$$

$$\therefore \lim_{N \to \infty} G_{m^A[\lambda(N)]}(z) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \frac{1}{z - \hat{f}(\frac{j}{N})} = \int_0^1 \frac{1}{z - \hat{f}(x)} dx \qquad \blacksquare$$

§ 6 Asymptotic analysis of Characters (only type A)

Goal 1: asymptotic analysis of characters

Thm 6.1 $\lambda(N) \in \widehat{G}_N$ $(N \geq 1)$: regular s.t. $m^G[\lambda(N)] \to m$

$$\forall k \geq 1, \quad \lim_{N\to\infty} \frac{1}{N} \log \frac{\chi^G_{\lambda(N)}(z_1, \dots, z_k, 1^{N-k})}{\chi^G_{\lambda(N)}(1^{N-k})} = H_m(z_1) + \dots + H_m(z_k)$$

uniformly in an open neighborhood of $(1^k) \in \mathbb{C}^k$,

where $H_m(z) = \int_0^{\ln(z)} R_m^{quant}(t)\,dt$

Goal 2: asymptotic analysis of probability measures from characters

Thm 6.2: $P_N \in \mathcal{P}(\widehat{U(N)})$ $(N \geq 1)$ s.t. $S^{U(N)}_{P_N}(z)$ is well-defined

Assume $\forall k \geq 1$, $\lim_{N\to\infty} \frac{1}{N} \log S^{U(N)}_{P_N}(z_1, \dots, z_k, 1^{N-k}) = Q(z_1) + \dots + Q(z_k)$

uniformly in an open neighborhood of $(1^k) \in \mathbb{C}^k$

where $Q(z)$ is analytic

Then $m^A[P_N] \to \exists m$ in the moments sense, in probability

$$\& \quad M_q(m) = \sum_{l=0}^{q} \frac{q!}{l!\,(l+1)!\,(q-l)!} \frac{d^l}{dz^l} \left(z^q Q'(z)^{q-l} \right)\Big|_{z=1}$$

◎ Consequences of Thm 6.1, 6.2

Lem 6.3 $m \in \mathcal{P}(\mathbb{R})$ compact support

$$\forall q \geq 1, \qquad M_q(m) = \sum_{l=0}^{q} \frac{q!}{l!\,(l+1)!\,(q-l)!} \frac{d^l}{dz^l}\left(z^{q-l} H_m'(z)^{q-l}\right)\Big|_{z=1}$$

Proof) $H_m(z) = \int_0^{\ln(z)} R_m^{quant}(t)\,dt \quad \rightsquigarrow \quad H_m'(z) = R_m^{quant}(\ln(z)) \cdot \frac{1}{z}$

Recall : $C_m(z) := G_m(z^{-1}) = \int \frac{z}{1-zx}\,dm(x) = \sum_{p=0}^{\infty} M_p(m) z^{p+1}$

$$\& \quad R_m^{quant}(z) = \frac{1}{C_m^{\langle -1 \rangle}(z)} - \frac{1}{1-e^{-z}}$$

$$\therefore \quad C_m^{\langle -1 \rangle}(z) = \frac{1}{R_m^{quant}(z) + \frac{1}{1-e^{-z}}}$$

$$= \frac{1}{e^z H_m'(e^z) + \frac{1}{1-e^{-z}}}$$

◎ The Lagrange inversion formula (see [Stanley's book, Thm 5.4.2])

$F(x) = a_1 x + a_2 x^2 + \cdots, \quad a_1 \neq 0 \quad \rightsquigarrow \quad F^{\langle -1 \rangle}(x)$: inverse

$$\Rightarrow \quad [x^n] F^{\langle -1 \rangle}(x) = \frac{1}{n} [x^{n-1}] \left(\frac{x}{F(x)}\right)^n$$

$$\therefore \quad M_q(m) = [z^{q+1}]\, C_m(z)$$

$$= \frac{1}{q+1} [z^q] \left(z^{q+1} \left(e^z H_m'(e^z) + \frac{1}{1-e^{-z}} \right)^{q+1} \right)$$

$$= \frac{1}{q+1} [z^{-1}] \left(e^z H_m'(e^z) + \frac{1}{1-e^{-z}} \right)^{q+1}$$

$$= \frac{1}{q+1} \cdot \frac{1}{2\pi i} \oint_0 \left(e^z H_m'(e^z) + \frac{1}{1-e^{-z}} \right)^{q+1} dz$$

$w = e^z \downarrow$

$$= \frac{1}{q+1} \frac{1}{2\pi i} \oint_1 \left(w H_m'(w) + \frac{w}{w-1} \right)^{q+1} \frac{1}{w}\, dw$$

$$= \frac{1}{q+1} \frac{1}{2\pi i} \oint_1 w^q \cdot \sum_{l=0}^{q+1} \binom{q+1}{l} H_m'(w)^{q+1-l} \frac{1}{(w-1)^l}\, dw$$

$\ell = 0 \Rightarrow \oint_1 \cdots = 0 \longrightarrow$
$$= \frac{1}{q+1} \sum_{\ell=0}^{q+1} \binom{q+1}{\ell} \frac{1}{2\pi i} \oint_1 w^q \cdot \frac{H_m'(w)^{q+1-\ell}}{(w-\ell)^\ell}\, dw$$

$$= \frac{1}{q+1} \sum_{\ell=0}^{q} \binom{q+1}{\ell+1} \frac{1}{2\pi i} \oint_1 w^q \cdot \frac{H_m'(w)^{q-\ell}}{(w-\ell)^{\ell+1}}\, dw$$

$$= \sum_{\ell=0}^{q} \frac{q!}{(\ell+1)!\,(q-\ell)!} \cdot \frac{1}{\ell!} \left(\frac{d}{dw} \right)^\ell w^q H_m'(w)^{q-\ell} \Big|_{w=1} \qquad \boxed{/\!/}$$

<u>Def</u> : $\lambda^1, \ldots, \lambda^k \in \widehat{U(N)}$,

$\underline{P^{\lambda^1 \otimes \cdots \otimes \lambda^k}} \in \mathcal{P}(\widehat{U(N)})$ associated with <u>the character of $\pi_{\lambda^1} \otimes \cdots \otimes \pi_{\lambda^k}$</u>

e.g. $p^{\lambda^1 \otimes \lambda^2}(\mu) = \dfrac{c^{\mu}_{\lambda_1, \lambda_2}\, d_\mu}{d_{\lambda_1} d_{\lambda_2}} \quad \left(\forall \mu \in \widehat{U(N)}\right) \qquad = \dfrac{\chi^A_{\lambda^1} \cdots \chi^A_{\lambda^k}}{d_{\lambda^1} \cdots d_{\lambda^k}}$

where $\pi_{\lambda_1} \otimes \pi_{\lambda_2} \sim \bigoplus_\mu \pi_\mu^{\oplus c^{\mu}_{\lambda_1, \lambda_2}}$

Thm : Asymptotic behavior of tensor product representations

$\lambda^i(N) \in \widehat{U(N)}$ $(N \geq 1,\ i = 1, \dots, k)$: regular s.t. $m^A[\lambda^i(N)] \to m^i$ weakly

$\Rightarrow$ (1) $m^A[p^{\lambda^1(N) \otimes \cdots \otimes \lambda^k(N)}] \longrightarrow m^1 \otimes \cdots \otimes m^k$

in the moment sense, in probability.

(2) $m^A_{pp}[p^{\lambda^1(N) \otimes \cdots \otimes \lambda^k(N)}] \longrightarrow Q(m^1) \boxplus \cdots \boxplus Q(m^k)$

$\not\longrightarrow$

Rem : By Thm 4.5, (2) follows from (1)

Proof) Since $S^{U(N)}_{p^{\lambda^1(N) \otimes \cdots \otimes \lambda^k(N)}}(\underline{z}) = S^{U(N)}_{p^{\lambda^1(N)}}(\underline{z}) \cdots S^{U(N)}_{p^{\lambda^k(N)}}(\underline{z})$,

by Thm 6.1, $\forall k \geq 1$,

$$\lim_{N \to \infty} \frac{1}{N} \log S^{U(N)}_{p^{\lambda^1(N) \otimes \cdots \otimes \lambda^k(N)}}(z_1, \dots, z_k, 1^{N-k})$$

$$= Q(z_1) + \cdots + Q(z_k),$$

where $Q(z) = H_{m_1}(z) + \cdots + H_{m_k}(z)$

$$= H_{m_1 \boxplus \cdots \boxplus m_k} \quad (2)$$

by Thm 6.2, Lem 6.3, (1) holds true $\square$

<u>Def</u>: $\lambda \in \widehat{U(N)}$, $0 < \alpha < 1$

$P^{\alpha, \lambda} \in \mathbb{P}(\widehat{U(\lfloor \alpha N \rfloor)})$ associated with $\chi^A_\lambda|_{U(\lfloor \alpha N \rfloor)}$

i.e., $$P^{\alpha,\lambda}(\mu) = \frac{m^\lambda_\mu d_\mu}{d_\lambda} \quad (\forall \mu \in \widehat{U(\lfloor \alpha N \rfloor)})$$

where $$\pi_\lambda|_{U(\lfloor \alpha N \rfloor)} \sim \bigoplus_\mu \pi_\mu^{\oplus m^\lambda_\mu}$$

<u>Thm</u>: Asymptotic behavior of restrictions

$\lambda(N) \in \widehat{U(N)}$: regular s.t. $m^A[\lambda(N)] \to m$ weakly

$\Rightarrow$ (1) $m^A[P^{\alpha, \lambda(N)}] \longrightarrow m^{\otimes \frac{1}{\alpha}}$ in the moment sense, in prob.

(2) $m^A_{pp}[P^{\alpha, \lambda(N)}] \longrightarrow Q(m)^{\boxplus \frac{1}{\alpha}}$ ——"——

where $$m^{\otimes \frac{1}{\alpha}} \longleftrightarrow R^{quant}_{m^{\otimes \frac{1}{\alpha}}} = \frac{1}{\alpha} R^{quant}_m$$

$$Q(m)^{\boxplus \frac{1}{\alpha}} \longleftrightarrow R_{Q(m)^{\boxplus \frac{1}{\alpha}}} = \frac{1}{\alpha} R_{Q(m)}$$

<u>Rem</u>. By Thm 4.5, (2) follows from (1)

Proof) Since $S^{U(\lfloor \alpha N \rfloor)}_{p\alpha, \lambda(N)}(\underline{z}) = \dfrac{\chi^A_{\lambda(N)}(\underline{z}, 1^{N-\lfloor \alpha N \rfloor})}{\chi^A_{\lambda(N)}(1^N)}$

by Thm 6.1, $\forall k \geq 1$,

$$\lim_{N\to\infty} \frac{1}{\lfloor \alpha N \rfloor} \log S^{U(\lfloor \alpha N \rfloor)}_{p\alpha, \lambda(N)}(z_1, \ldots, z_k, 1^{N-k})$$

$$= \frac{1}{\alpha}\left(H_m(z_1) + \cdots + H_m(z_k) \right)$$

$$= H_{m \otimes \frac{1}{\alpha}}(z_1) + \cdots + H_{m \otimes \frac{1}{\alpha}}(z_k)$$

$\rightsquigarrow$ by Thm 6.2, Lem 6.3, (1) holds true ▨

<u>Rem</u>: The above two theorems hold for $G \neq A$.

∘ Proof of Thm 6.1 (Outline)

1. expression of $\dfrac{\chi^A_\lambda(x_1, \ldots, x_k, 1^{N-k})}{\chi^A_\lambda(1^N)}$ by $\dfrac{\chi^A_\lambda(x_i, 1^{N-1})}{\chi^A_\lambda(1^N)}$

2. asymptotic behavior of $\dfrac{\chi^A_\lambda(x, 1^{N-1})}{\chi^A_\lambda(1^N)}$

1. Def. $\mu = (\mu_1 > \dots > \mu_N) \in \mathbb{Z}^N$,

$$A_\mu(x_1, \dots, x_N) := \frac{\det\left[x_i^{\mu_j}\right]_{i,j=1}^{N}}{\det\left[x_i^{N-j}\right]_{i,j=1}^{N}}$$

i.e., $\forall \lambda \in \widehat{U(N)}$, $\mu = (\lambda_1 + N-1, \lambda_2 + N-2, \dots, \lambda_N) \rightarrow A_\mu = \chi_\lambda^A$

Lem 6.4 $A_\mu(x_1, \dots, x_k, 1^{N-k})$

$$= \frac{(-1)^{\binom{k}{2}}}{\prod_{i=1} (x_i - 1)^{N-k}} \sum_{\substack{J \subset \{1,\dots,N\} \\ |J| = k}} (-1)^{\sum_{j \in J}(j-1)} A_{\underbrace{\mu|J}_{(\mu_j)_{j\in J}}}(\underline{x})\, A_{\mu|J^c}(1^{N-k})$$

proof) $A_\mu(x_1, \dots, x_k, 1, q, \dots, q^{N-k-1})$

$$= \frac{\det\left[\ \cdots \dfrac{x_i^{\mu_j}}{q^{(i-1)\mu_j}} \cdots\ \right]_{i,j=1}^{N}}{\prod_{1 \le i < j \le k} (x_i - x_j) \cdot \prod_{\substack{i=1,\dots,k \\ j=1,\dots,N-k}} (x_i - q^{j-1}) \cdot \prod_{1 \le i < j \le N-k} (q^{i-1} - q^{j-1})}$$

$\because$ $\det[a_{ij}]_{i,j=1}^{N} = (-1)^{\binom{k}{2}} \sum_{J \subset \{1\dots N\}} (-1)^{\sum_{j \in J}(j-1)} \det[a_{ij}]_{\substack{i=1,\dots,k \\ j \in J}} \det[a_{ij}]_{\substack{i=k+1,\dots,N \\ j \in J^c}}$

$$|J|=k$$

$$\rightsquigarrow \quad = \frac{(-1)^{\binom{k}{2}} \sum_J (-1)^{\sum_{j\in J}(j-1)} \cdot \det\left[x_i^{\mu_j} \right]_{\substack{i=1,\dots,k \\ j\in J}} \det\left[q^{(i-1)\mu_j} \right]_{\substack{i=1,\dots,N-k \\ j\in J^c}}}{\prod_{1\le i<j\le k} (x_i-x_j) \cdot \prod_{\substack{i=1,\dots,k \\ j=1,\dots,N-k}} (x_i-q^{j-1}) \cdot \prod_{1\le i<j\le N-k} (q^{i-1}-q^{j-1})}$$

$$= \frac{(-1)^{\binom{k}{2}}}{\prod_{\substack{i=1,\dots,k, \\ j=1,\dots,N-k}} (x_i-q^{j-1})} \sum_J (-1)^{\sum_{j\in J}(j-1)} A_{\mu|_J}(\underline{x})\, A_{\mu|_{J^c}}(1,\dots,q^{N-k-1})$$

$\rightsquigarrow$ the assertion follows by $q \to 1$ ▨

$$\therefore \quad \frac{A_\mu(\underline{x}, 1^{N-k})}{A_\mu(1^N)} = \frac{(-1)^{\binom{k}{2}}}{\prod_{i=1}^{k} (x_i-1)^{N-k}} \sum_J (-1)^{\sum_{j\in J}(j-1)} A_{\mu|_J}(\underline{x})\, \underbrace{\frac{A_{\mu|_{J^c}}(1^{N-k})}{A_\mu(1^N)}}_{\text{the Weyl dimension formula}}$$

$$= \frac{C_{N-k}}{C_N} \cdot \frac{\prod_{1\le i<j\le N-k} \left((\mu|_{J^c})_i - (\mu|_{J^c})_j \right)}{\prod_{i<j} (\mu_i-\mu_j)}$$

where $C_k := \prod_{1\le i<j\le k} \frac{1}{j-i}$

$$
= \frac{C_{N-k}}{C_N} \cdot \frac{1}{\underbrace{\prod_{i<j \text{ in } J} \times \prod_{\substack{i<j \\ i\in J \\ j\in J^c}}}_{i<j,\ i\in J,} \times \underbrace{\prod_{\substack{i<j \\ i\in J^c \\ j\in J}} (\mu_i-\mu_j)}_{\frac{\prod_{i<j \text{ in } J}(\mu_i-\mu_j)}{\prod_{\substack{i<j \\ j\in J}}(\mu_i-\mu_j)}}}
$$

$$
= \frac{C_{N-k}}{C_N} \cdot \frac{\prod_{i<j \text{ in } J} (\mu_i-\mu_j)}{\prod_{i\in J} (-1)^{i-1} \prod_{j\neq i} (\mu_i-\mu_j)}
$$

$$
\therefore \quad \frac{A_\mu(x, 1^{N-k})}{A_\mu(1^N)} = \frac{C_{N-k}}{C_N} \frac{(-1)^{\binom{k}{2}}}{\prod_{i=1}^{k} (x_i-1)^{N-k}} \sum_J \underline{A_{\mu|J}(x)} \frac{\underline{\prod_{i<j \text{ in } J} (\mu_i-\mu_j)}}{\prod_{i\in J} \prod_{j\neq i} (\mu_i-\mu_j)} \quad \text{—(6.5)}
$$

Lem: $\forall \mu = (\mu_1 > \cdots > \mu_k)$,

$$
\prod_{1\le i<j\le k} (\mu_i-\mu_j) \cdot A_\mu(x) = \frac{(-1)^{\binom{k}{2}}}{VA(x)} \det\left[\underset{\substack{\uparrow \\ x_i \frac{\partial}{\partial x_i}}}{E}{}_{x_j}^{\,i-1} \right]_{i,j=1}^{k} \sum_{\sigma\in S_k} \prod_{j=1}^{k} x_j^{\mu_{\sigma(j)}}
$$

Proof)

$$\det\left[E_{x_j}^{i-1}\right]_{i,j=1}^{k} \sum_{\sigma} \prod_{j=1}^{k} x_j^{\mu_{\sigma(j)}}$$

$$= \sum_{\sigma} \det\left[E_{x_j}^{i-1} \cdot x_j^{\mu_{\sigma(j)}}\right]$$

$$= \sum_{\sigma} \det\left[\mu_{\sigma(j)}^{i-1} \cdot x_j^{\mu_{\sigma(j)}}\right]$$

$$= \det\left[\mu_j^{i-1}\right] \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{j=1}^{k} x_{\sigma^{-1}(j)}^{\mu_j}$$

$$= (-1)^{\binom{k}{2}} \cdot \prod_{1\le i<j\le k} (\mu_i - \mu_j) \cdot \det\left[x_i^{\mu_j}\right]_{i,j=1}^{k} \qquad \blacksquare$$

$$\therefore \frac{A_\mu(x, 1^{N-k})}{A_\mu(1^N)}$$

$$= \frac{C_{N-k}}{C_N} \frac{1}{\prod_{i=1}^{k}(x_i-1)^{N-k}} \sum_{J} \frac{1}{\prod_{i\in J}\prod_{j\neq i}(\mu_i-\mu_j)} \frac{\det\left[E_{x_j}^{i-1}\right]_{i,j=1}^{k}}{VA(x)} \sum_{\sigma\in S_k} \prod_{j=1}^{k} x_j^{(\mu|_J)_{\sigma(j)}}$$

$$= \frac{C_{N-k}}{C_N} \frac{1}{\prod_{i=1}^{k}(x_i-1)^{N-k}} \frac{\det\left[E_{x_j}^{i-1}\right]_{i,j=1}^{k}}{VA(x)} \underbrace{\sum_{J} \sum_{\sigma\in S_k}}_{\substack{j_1,\dots,j_k=1,\dots,N \\ (j_a\neq j_b)}} \prod_{j=1}^{k} \frac{x_j^{(\mu|_J)_{\sigma(j)}}}{\prod_{i\neq\sigma(j)}\left((\mu|_J)_{\sigma(j)} - \mu_i\right)}$$

$$\text{※} \quad \det\left[E_{x_j}^{i-1}\right]_{i,j=1}^{k} = \prod_{1\le i<j\le k}\left(E_{x_j} - E_{x_i}\right)$$

$j_1, \dots, j_k = 1, \dots, N$

$(j_a \neq j_b)$

$$\rightsquigarrow \quad \det\left[E_{x_j}^{i-1}\right]_{i,j=1}^{k} \cdots x_i^m \cdots x_j^m \cdots = 0$$

$$\therefore \quad = \frac{C_{N-k}}{C_N} \frac{1}{\prod_{i=1}^{k}(x_i-1)^{N-k}} \frac{\det\left[E_{x_j}^{i-1}\right]_{i,j=1}^{k}}{V^A(\underline{x})} \cdot \sum_{j_1,\dots,j_k=1}^{N} \prod_{r=1}^{k} \frac{x_r^{\mu_{j_r}}}{\prod_{i\neq j_r}(\mu_{j_r}-\mu_i)}$$

$$= \frac{C_{N-k}}{C_N} \cdot \frac{1}{\prod_{i=1}^{k}(x_i-1)^{N-k}} \frac{\det\left[E_{x_j}^{i-1}\right]_{i,j=1}^{k}}{V^A(\underline{x})} \prod_{r=1}^{k} \sum_{j=1}^{N} \frac{x_r^{\mu_j}}{\prod_{i\neq j}(\mu_j-\mu_i)}$$

Lem 6.6

$$\frac{A_\mu(x,1^{N-1})}{A_\mu(1^N)} = \frac{(N-1)!}{(x-1)^{N-1}} \sum_{j=1}^{N} \frac{x^{\mu_j}}{\prod_{i\neq j}(\mu_j-\mu_i)}$$

proof) By Eq. (6.5)

$$(\mathrm{LHS}) = \underbrace{\frac{C_{N-1}}{C_N}}_{(N-1)!} \frac{1}{(x-1)^{N-1}} \sum_{j=1}^{N} \underbrace{A_{\mu_j}(x)}_{x^{\mu_j}} \cdot \frac{1}{\prod_{i\neq j}(\mu_j-\mu_i)}$$

∎

$$\therefore \quad \frac{A_\mu(x,1^{N-k})}{A_\mu(1^N)} = \frac{C_{N-k}}{C_N} \frac{1}{\prod_{i=1}^{k}(x_i-1)^{N-k}} \cdot \frac{\det\left[E_{x_j}^{i-1}\right]_{i,j=1}^{k}}{V^A(\underline{x})} \prod_{r=1}^{k} \frac{(x_r-1)^{N-1}}{(N-1)!} \frac{A_\mu(x_r,1^{N-1})}{A_\mu(1^N)}$$

$$= \prod_{i=1}^{k} \frac{(N-i)!}{(N-1)!\,(x_i-1)^{N-k}} \frac{\det\left[E_{x_j}^{i-1}\right]_{i,j=1}^{k}}{V^A(\underline{x})} \cdot \prod_{r=1}^{k} (x_r-1)^{N-1} \frac{A_\mu(x_r, 1^{N-1})}{A_\mu(1^N)}$$

(6.7) $$= \frac{1}{V^A(\underline{x})} \det\left[\frac{(N-i)!}{(N-1)!} \frac{1}{(x_j-1)^{N-k}} E_{x_j}^{i-1} \cdot (x_j-1)^{N-1} \frac{A_\mu(x_j, 1^{N-1})}{A_\mu(1^N)} \right]_{i,j=1}^{k}$$

2.

Prop $$\frac{\chi_\lambda^A(x, 1^{N-1})}{\chi_\lambda^A(1^N)} = \frac{(N-1)!}{N^{N-1}(x-1)^{N-1}} \frac{1}{2\pi i} \oint_C \frac{x^{NW}}{\prod_{j=1}^{N}(w - \Delta_j/N)} dw$$

where C contains $\frac{\Delta_j}{N} = \frac{\lambda_j + N - j}{N}$ $(j=1, \dots, N)$

Proof) By Lem. 6.6, $\boxed{///}$

$$\frac{\chi_\lambda^A(x, 1^{N-1})}{\chi_\lambda^A(1^N)} = \frac{(N-1)!}{(x-1)^{N-1}} \frac{1}{2\pi i} \oint \frac{x^z}{\prod_{j=1}^{N}(z - \Delta_j)} dz \rightsquigarrow z = Nw$$

Contains $\Delta_1, \dots, \Delta_N$ $\boxed{///}$

$$\therefore \frac{1}{N} \log \frac{\chi_\lambda^A(x, 1^{N-1})}{\chi_\lambda^A(1^N)} = \underbrace{\frac{1}{N} \log \frac{(N-1)!}{N^{N-1}(x-1)^{N-1}}}_{①} + \underbrace{\frac{1}{N} \log \frac{1}{2\pi i} \oint_C \frac{x^{NW}}{\prod_{j=1}^{N}\left(w - \frac{\Delta_j}{N}\right)} dw}_{②}$$

① $= \frac{1}{N} \log \frac{(N-1)!}{x^{N-1}} - \frac{N-1}{N} \log (x-1)$

$$\sim -\log(x-1) + \frac{1}{N} \log \frac{\sqrt{2\pi(N-1)}\,(N-1)^{N-1} \cdot e^{-(N-1)}}{N^{N-1}} \qquad \text{(by the Stirling approximation)}$$

$$\sim -\log(x-1) - 1$$

② $x = e^{y}$

$$\frac{1}{2\pi i} \int_{\tau} \frac{e^{Ny \cdot w}}{\prod_{j=1}^{N} (w - \frac{\lambda_j}{N})} dw = \frac{1}{2\pi i} \int_{\tau} \exp\left\{ N\left(yw - \frac{1}{N} \sum_{j=1}^{N} \log\left(w - \frac{\lambda_j}{N}\right)\right)\right\} dw$$

<u>assume</u> : $\lambda(N) \in \widehat{U(N)}$: regular s.t $\hat{m}[\lambda(N)] \longrightarrow m$

$\rightsquigarrow$ recall : $G_m(z) = \int_0^1 \frac{1}{z - \hat{f}(x)} dx \qquad (\hat{f}(x) = f(x) - x + 1)$

$\rightsquigarrow \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \log\left(w - \frac{\lambda_j}{N}\right) = \int_0^1 \log(w - \hat{f}(x))\, dx =: \mathcal{F}(w)$

$$\rightsquigarrow = \frac{1}{2\pi i} \int_{\tau} \underbrace{\exp\left\{ N\left(yw - \mathcal{F}(w)\right)\right\}}_{\text{main term}} \underbrace{\exp\left\{ N\left(\mathcal{F}(w) - \frac{1}{N} \sum_{j=1}^{N} \log\left(w - \frac{\lambda_j}{N}\right)\right)\right\}}_{\text{error term}} dw$$

Saddle point method

$$\sim \quad \exp\{ N (y w_0 - \mathcal{F}(w_0)) \}$$

where w_0 solves $(y w - \mathcal{F}(w))' = 0$

i.e., $$y = \mathcal{F}'(w_0) = \int_0^1 \frac{1}{w_0 - \hat{f}(x)} dx = G_m(w_0)$$

$$\therefore \quad w_0 = R_m^{quant}(y) + \frac{1}{1-e^{-y}}$$

$$\therefore \quad \frac{1}{N} \log \frac{\chi^A_{\lambda(N)}(e^y, 1^{N-1})}{\chi^A_{\lambda(N)}(1^N)}$$

$$\sim \quad -\log(e^y - 1) - 1 + y w_0 - \mathcal{F}(w_0)$$

<u>Rem</u> $$\left(-\log(e^y - 1) - 1 + y w_0 - \mathcal{F}(w_0) \right)'$$

$$= -\frac{e^y}{e^y - 1} + w_0 + y w_0' - \mathcal{F}'(w_0) w_0' \qquad \because \; y = \mathcal{F}'(w_0)$$

$$= R_m^{quant}(y)$$

$$\therefore \quad \frac{1}{N} \log \frac{\chi^A_{\lambda(N)}(e^y, 1^{N-1})}{\chi^A_{\lambda(N)}(1^N)} \sim \underbrace{\int_0^y R_m^{quant}(t)\, dt}_{H_m(e^y)} + C$$

$C = 0$ taking $y \to 0$

Moreover, ...

Prop 6.8 $\lambda(N) \in \widehat{U(N)}$ $(N \geq 1)$: regular s.t. $m^A[\lambda(N)] \to m$

$$\Longrightarrow \lim_{N\to\infty} \frac{1}{N} \log \frac{\chi^A_{\lambda(N)}(x, 1^{N-1})}{\chi^A_{\lambda(N)}(1^N)} = H_m(x)$$ uniformly in an open n.b. of $x=1$

proof of Thm 6.7)

Set $$\tilde{\chi}^A_{\lambda(N)}(x_1, \dots, x_k, 1^{N-k}) := \frac{\chi^A_{\lambda(N)}(x_1, \dots, x_k, 1^{N-k})}{\chi^A_{\lambda(N)}(1^N)}$$

By Eq (6.7),

$$\frac{\tilde{\chi}^A_{\lambda(N)}(x_1, \dots, x_k, 1^{N-k})}{\prod_{j=1}^{N} \tilde{\chi}^A_{\lambda(N)}(x_j, 1^{N-1})}$$

$$= \frac{1}{V^A(\underline{x})} \det\left[\frac{(N-i)!}{(N-1)!} \underbrace{\frac{1}{(x_j-1)^{N-k}} \frac{E^{i-1}_{x_j} (x_j-1)^{N-1} \tilde{\chi}^A_{\lambda(N)}(x_j, 1^{N-1})}{\tilde{\chi}^A_{\lambda(N)}(x_j, 1^{N-1})}} \right]^k_{i,j=1}$$

$$\sum_{l=0}^{i-1} \binom{i-1}{l} \underbrace{\frac{E_{x_j}^{l}\, \tilde{\chi}^{A}_{\lambda(N)}(x_j, 1^{N-1})}{\tilde{\chi}^{A}_{\lambda(N)}(x_j, 1^{N-1})}} \; \underbrace{\frac{E_{x_j}^{i-1-l}\, (x_j-1)^{N-1}}{(x_j-1)^{N-k}}}_{\star}$$

linear combination of $x_j^m \underbrace{\dfrac{\left(\frac{d}{dx_j}\right)^m \tilde{\chi}^{A}_{\lambda(N)}(x_j, 1^{N-1})}{\tilde{\chi}^{A}_{\lambda(N)}(x_j, 1^{N})}}$ $(m = 0, \dots, l)$

Polynomial in $\left(\frac{d}{dx_j}\right)^s \log \tilde{\chi}^{A}_{\lambda(N)}(x_j, 1^{N-1})$ of degree m

by Prop 6.8 $\longrightarrow \; = O(N^m)$

$$\star = (N-1)^{i-1-l} \frac{x_j^{i-1-l} (x_j-1)^{N-i+l}}{(x_j-1)^{N-k}} + O(N^{i-1-l})$$

$$= O(N^{i-1})$$

$$\rightsquigarrow \frac{\tilde{\chi}^{A}_{\lambda(N)}(x_1, \dots, x_k, 1^{N-k})}{\prod_{j=1}^{k} \tilde{\chi}^{A}_{\lambda(N)}(x_j, 1^{N-k})}$$ Converges a bounded function

$$\therefore \frac{1}{N} \log \widetilde{\chi}^{A}_{\lambda(N)}(x_1, \dots, x_k, 1^{N-k})$$

$$\sim \frac{1}{N} \log \prod_{j=1}^{k} \widehat{\chi}^{A}_{\lambda(N)}(x_j, 1^{N-1}) \longrightarrow H_m(x_1) + \dots + H_m(x_k) \qquad \blacksquare$$

◦ Proof of Thm 6.2

Assume $\quad \frac{1}{N} \log S^{G_N}_{P_N}(x_1, \dots, x_k, 1^{N-k}) \to Q(x_1) + \dots + Q(x_k)$

Then $\quad \forall q \geq 1, \quad M_q(m^A[P_N])$

$$\longrightarrow M_q(m) = \sum_{l=0}^{q} \frac{q!}{l!(l+1)!(q-l)!} \left(\frac{d}{dz}\right)^{l} \left(z^q Q'(z)^{q-l}\right)\Big|_{z=1}$$

in probability

<u>Strategy</u>: $\quad X_N \ (N \geq 1)$: random variables,

s.t. $\quad \mathbb{E}[X_N] \to a, \quad \mathbb{E}[X_N^2] \to a^2$

$\Rightarrow \quad X_N \to a \quad$ in probability

$\because$) By the Chebyshev inequality,

$$\forall \varepsilon > 0, \quad \mathbb{P}(|X_N - a| > \varepsilon) \leq \frac{1}{\varepsilon^2} \mathbb{E}[|X_N - a|^2]$$

$$= \frac{1}{\varepsilon^2}\left(\mathbb{E}[X_N^2] - 2a\,\mathbb{E}[X_N] + a^2\right)$$

$$\longrightarrow 0 \quad \text{as } N \to \infty \qquad //$$

$\rightsquigarrow$ Our goal : limit of $\mathbb{E}[M_q(m^A[P_N])^m]$ $(m = 1, 2)$

<u>Recall</u> : By Prop 4. X ,

$$\mathbb{E}[M_q(m^A[P_N])^m] = \left.\frac{(\mathcal{D}_q^{U(N)})^m S_{P_N}^{U(N)}(\underline{x})}{S_{P_N}^{U(N)}(\underline{x})}\right|_{\underline{x} = (1^N)}$$

where $\mathcal{D}_q^{U(N)} = \frac{1}{V^A(\underline{x})} \sum_{i=1}^{N} E_{x_i}^q \cdot V^A(\underline{x})$

By assumption,

$$S_{P_N}^{U(N)}(\underline{x}) = e^{N \sum_{i=1}^N Q(x_i)} \cdot T_N(\underline{x}) ,$$

where $Q(1) = 0$, $T_N(\underline{x})$: analytic , symmetric

& $\frac{1}{N} \log T_N(x_1, \ldots, x_k, 1^{N-k}) \longrightarrow 0$

$\rightsquigarrow$ $(\mathcal{D}_q^{U(N)})^m S_{P_N}^{U(N)}(\underline{x})$

$$= \frac{1}{V^A(\underline{x})} \sum_{i_1, \ldots, i_m = 1}^{N} E_{x_{i_1}}^q \cdots E_{x_{i_m}}^q \cdot V^A(\underline{x}) \cdot e^{N \sum_{i=1}^N Q(x_i)} \cdot T_N(\underline{x})$$

$$= \frac{1}{V^A(\underline{x})} \sum_{s=1}^{m} \sum_{i_1<\cdots<i_s} \sum_{\substack{m_1+\cdots+m_s=m \\ m_i\geq 1}} \underbrace{\frac{m!}{m_1!\cdots m_s!}} E_{x_{i_1}}^{q m_1}\cdots E_{x_{i_s}}^{q m_s} \text{---}$$

$$= \frac{1}{V^A(\underline{x})} \sum_{s=1}^{m} \sum_{\underline{i}} \sum_{\underline{m}} \binom{m}{\underline{m}} \sum_{\substack{a_r+b_r+c_r=q m_r \\ (r=1,\ldots,s)}} \underbrace{\prod_r \frac{(q m_r)!}{a_r!\, b_r!\, c_r!}}$$

$$\times E_{x_{\underline{i}}}^{\underline{a}} V^A(\underline{x})\ E_{x_{\underline{i}}}^{\underline{b}} e^{N\sum_i Q(x_i)} \cdot E_{x_{\underline{i}}}^{\underline{c}} T_N(\underline{x})$$

$$\rightsquigarrow \frac{(\mathcal{D}_q^{U(N)})^m S_{P_N}^{U(N)}(\underline{x})}{S_{P_N}^{U(N)}(\underline{x})}$$

$$= \sum_{s,\underline{i}} \sum_{\underline{m}} \sum_{\underline{a},\underline{b},\underline{c}} \binom{m}{\underline{m}}\, C\, \frac{E_{x_{\underline{i}}}^{\underline{a}} V^A(\underline{x})}{V^A(\underline{x})} \cdot \frac{E_{x_{\underline{i}}}^{\underline{b}} e^{N\sum_i Q(x_i)}}{e^{N\sum_i Q(x_i)}} \cdot \frac{E_{x_{\underline{i}}}^{\underline{c}} T_N(\underline{x})}{T_N(\underline{x})}$$

$\boxed{m=1}$

$$\frac{\mathcal{D}_q^{U(N)} S_{P_N}^{U(N)}(\underline{x})}{S_{P_N}^{U(N)}(\underline{x})}$$

$s=1$

$$= \sum_{i=1}^{N} \sum_{a+b+c=q} \frac{q!}{a!\,b!\,c!} \underbrace{\frac{E_{x_i}^{a} V^A(\underline{x})}{V^A(\underline{x})}} \cdot \frac{E_{x_i}^{b} e^{N\sum_i Q(x_i)}}{e^{N\sum_i Q(x_i)}} \cdot \frac{E_{x_i}^{c} T_N(\underline{x})}{T_N(\underline{x})}$$

$$
\frac{E^a_{x_i} V^A(\underline{x})}{V^A(\underline{x})} = \frac{E^a_{x_i} \prod_{j \neq i} (x_i - x_j)}{\prod_{j \neq i} (x_i - x_j)}
$$

$$
= \sum_{k=1}^{a} \sum_{j_1 < \cdots < j_k} \sum_{\substack{l_1 + \cdots + l_k = a \\ l_r \geq 1}} \frac{a!}{l_1! \cdots l_k!} \prod_{r=1}^{k} \frac{\overbrace{E^{l_r}_{x_i} (x_i - x_{j_r})}^{= x_i}}{x_i - x_{j_r}}
$$

$$
= \sum_{k, \underline{j}} \sum_{\underline{l}} \frac{a!}{l_1! \cdots l_k!} \frac{x_i^k}{\prod_{r=1}^{k} (x_i - x_{j_r})}
$$

$$
= \sum_{i=1}^{N} \sum_{a+b+c=q} \sum_{k, \underline{j}} \sum_{\underline{l}} \frac{q!}{b!\, c!\, l_1! \cdots l_k!} \frac{x_i^k}{\prod_{r=1}^{k} (x_i - x_{j_r})} \frac{E^b_{x_i} e^{NQ(x_i)}}{e^{NQ(x_i)}} \frac{E^c_{x_i} T_N(\underline{x})}{T_N(\underline{x})}
$$

$$
= \sum_{a+b+c=q} \sum_{k} \sum_{\underline{l}} \frac{q!}{b!\, c!\, l_1! \cdots l_k!} \sum_{\substack{j_1 < \cdots < j_{k+1} \\ k = 1, \ldots, a}} \sum_{r=1}^{k+1} \frac{x_{j_r}^k}{\prod_{s \neq r} (x_{j_r} - x_{j_s})} \frac{E^b_{x_{j_r}} e^{NQ(x_{j_r})}}{e^{NQ(x_{j_r})}} \times \frac{E^c_{x_{j_r}} T_N(\underline{x})}{T_N(\underline{x})}
$$

$$
\xrightarrow[\underline{x} \to (1^N)]{} \sum_{a+b+c=q} \sum_{k} \sum_{\underline{l}} \frac{q!}{b!\, c!\, \underline{l}!} \sum_{\underline{j}} \lim_{x_{\underline{j}} \to (1^{k+1})} \sum_{r=1}^{k+1} \text{''} \times \frac{E^c_{x_j} T_N(x_{\underline{j}}, 1^{N-k-1})}{T_N(x_{\underline{j}}, 1^{N-k-1})}
$$

Lem 6.9 : $f_r(x_1,\dots,x_n)$: analytic in a neighborhood of (1^n)

$(r=1,\dots,n)$

s.t. $\partial_{x_r}^{\ell} f_r(1^n) = \partial_{x_1}^{\ell} f_1(1^n)$ $(r=1,\dots,n)$ $\forall \ell \geq 1$

$$\Rightarrow \lim_{x\to(1^n)} \sum_{r=1}^{n} \frac{f_r(x_1,\dots,x_n)}{\prod_{s\neq r}(x_r-x_s)} = \frac{1}{(n-1)!}\partial_{x_1}^{n-1} f_1(1^n)$$

Applying Lem 6.9 for $f_r(x_{\underline{j}}) = x_{i_r}^k \frac{E_{x_{i_r}}^b e^{NQ(x_{j_r})}}{e^{NQ(x_{j_r})}} \frac{E_{x_j}^c T_N(x_{\underline{j}}, 1^{N-k-1})}{T_N(x_{\underline{j}}, 1^{N-k-1})}$

$$\to = \sum_{a+b+c=q} \sum_{k} \sum_{\underline{\ell}} \frac{q!}{b!\,c!\,\underline{\ell}!} \underbrace{\sum_{\underline{j}}}_{\binom{N}{k+1} = \frac{N^{k+1}}{(k+1)!} + O(N^k)} \frac{1}{k!}\left(\frac{d}{dx}\right)^k x^k \underbrace{\frac{E_x^b e^{NQ(x)}}{e^{NQ(x)}}} \cdot \underbrace{\frac{E_x^c T_N(x, 1^{N-1})}{T_N(x, 1^{N-1})}}\Bigg|_{x=1}$$

$x^{k+b} \cdot N^b Q'(x)^b$
$+ O(N^{b-1})$

Polynomial in x
& $\left(\frac{d}{dx}\right)^m \log T_N(x, 1^{N-1})$
of degree c
$= o(N^c)$
$\rightsquigarrow \boxed{c=0}$

$$\sim \sum_{a+b=q} \sum_{k} \sum_{\underline{\ell}} \frac{q!}{b!\,\underline{\ell}!} \frac{N^{k+1}}{(k+1)!} \frac{1}{k!} \left(\frac{d}{dx}\right)^k x^{k+b} N^b Q'(x)^b$$

$\boxed{k=a}$

$$\sim \sum_{a+b=q} \sum_{\underbrace{\substack{l_1+\cdots+l_a=a \\ l_r \geq 1}}} \frac{q!}{b!\, \underline{l}!} \frac{N^{a+b+1}}{(a+1)!} \cdot \frac{1}{a!} \left(\frac{d}{dx}\right)^a x^{a+b} \cdot Q'(x)^b$$

$\underline{l} = (1^a)$

$$= N^{q+1} \sum_{a=0}^{q} \frac{q!}{(q-a)!\,(a+1)!\,a!} \left(\frac{d}{dx}\right)^a x^q \cdot Q'(x)^{q-a}$$

$$\therefore \quad \mathbb{E}\left[M_q(m^A[P_N]) \right]$$

$$= \frac{1}{N^{q+1}} \frac{\mathcal{D}_q^{U(N)} S_{P_N}^{U(N)}(\underline{x})}{S_{P_N}^{U(N)}(\underline{x})} \Bigg|_{\underline{x}=(1^N)}$$

$$= \sum_{a=0}^{q} \frac{q!}{(q-a)!\,(a+1)!\,a!} \left(\frac{d}{dx}\right)^a x^q \cdot Q'(x)^{q-a}$$

$\boxed{m=2}$

$$\frac{\left(\mathcal{D}_q^{U(N)}\right)^2 S_{P_N}^{U(N)}(\underline{x})}{S_{P_N}^{U(N)}(\underline{x})}$$

leading term

$= \text{diagonal} + \text{off-diagonal}$

$$= \sum_{i_1<i_2} \sum_{\substack{a_r+b_r+c_r=q \\ (r=1,2)}} \prod_{r=1,2} \frac{q!}{a_r!\, b_r!\, c_r!} \frac{E_{x_{i_1}}^{a_1} E_{x_{i_2}}^{a_2} V^A(\underline{x})}{V^A(\underline{x})} \frac{E_{x_{i_1}}^{b_1} E_{x_{i_2}}^{b_2} e^{NQ(x_{i_1})+NQ(x_{i_2})}}{e^{NQ(x_{i_1})+NQ(x_{i_2})}}$$

$$\times \frac{E_{x_{i_1}}^{c_1} E_{x_{i_2}}^{c_2} T_N(\underline{x})}{T_N(\underline{x})}$$

$$\frac{E_{x_{i_1}}^{a_1} E_{x_{i_2}}^{a_2} (x_{i_1}-x_{i_2}) \prod_{j\neq i_1,i_2} (x_{i_1}-x_j)(x_{i_2}-x_j)}{(x_{i_1}-x_{i_2}) \prod_{j\neq i_1,i_2} (x_{i_1}-x_j)(x_{i_2}-x_j)}$$

$$= \sum_{\substack{k_r=1 \\ (r=1,2)}}^{a_r} \sum_{\substack{j_1^r<\cdots<j_{k_r}^r \\ (r=1,2)}} \sum_{\substack{l_0^r+\cdots+l_{k_r}^r=a_r \\ (r=1,2) \\ l_t^r\geq 1\ (t\geq 1)}} \prod_{r=1,2} \frac{a_r!}{l^r!} \frac{E_{x_{i_1}}^{l_0^1} E_{x_{i_2}}^{l_0^2} (x_{i_1}-x_{i_2})}{x_{i_1}-x_{i_2}}$$

$$\times \prod_{r=1,2} \prod_{t=1}^{k_r} \underbrace{\frac{E_{x_{i_r}}^{l_t^r} (x_{i_r}-x_{j_t^r})}{x_{i_r}-x_{j_t^r}}}_{= x_{i_r}}$$

まとめる

$$= \sum_{i_1<i_2} \sum_{\substack{a_r+b_r+c_r=q \\ (r=1,2)}} \sum_{k_r} \sum_{j^r} \sum_{l^r} \prod_{r=1,2} \frac{q!}{b_r!\, c_r!\, l^r!}$$

$$\frac{E_{x_{i_1}}^{l_0^1} E_{x_{i_2}}^{l_0^2} (x_{i_1}-x_{i_2})}{x_{i_1}-x_{i_2}} \prod_{r=1,2} \prod_{t=1}^{k_r} \frac{x_{i_r}}{x_{i_r}-x_{j_t^r}} \frac{E_{x_{i_1}}^{b_1} E_{x_{i_2}}^{b_2} e^{NQ(x_{i_1})+NQ(x_{i_2})}}{e^{NQ(x_{i_1})+NQ(x_{i_2})}}$$

$$\times \frac{E^{c_1}_{x_{i_1}} E^{c_2}_{x_{i_2}} T_N(\underline{x})}{T_N(\underline{x})}$$

$$= \sum_{a_r+b_r+c_r=q} \sum_{k_r} \sum_{\underline{l}^r} \prod_{r=1,2} \frac{q!}{b_r!\, c_r!\, \underline{l}^r!} \sum_{j^r_1 < \cdots < j^r_{k_r+1}}$$

$$\times \sum_{\substack{a_r = 1, \ldots, k_r+1 \\ j^1_{a_1} < j^2_{a_2}}} \frac{E^{l^1_0}_{x_{j^1_{a_1}}} E^{l^2_0}_{x_{j^2_{a_2}}} (x_{j^1_{a_1}} - x_{j^2_{a_2}})}{x_{j^1_{a_1}} - x_{j^2_{a_2}}} \prod_{r=1,2} \prod_{t \neq a_r} \frac{x_{j^r_{a_r}}}{x_{j^r_{a_r}} - x_{j^r_t}}$$

$$\times \frac{E^{b_1}_{x_{j^1_{a_1}}} E^{b_2}_{x_{j^2_{a_2}}} e^{NQ(x_{j^1_{a_1}}) + NQ(x_{j^2_{a_2}})}}{e^{NQ(x_{j^1_{a_1}}) + NQ(x_{j^2_{a_2}})}} \frac{E^{c_1}_{x_{j^1_{a_1}}} E^{c_2}_{x_{j^2_{a_2}}} T_N(\underline{x})}{T_N(\underline{x})}$$

$$\sum_{a=1}^{k_1+1} \sum_{\substack{b=1 \\ j^1_a < j^2_b}}^{k_2+1}$$

$$= \sum_{a_r+b_r+c_r=q} \sum_{k_r} \sum_{\underline{l}^r} \prod_{r=1,2} \frac{q!}{b_r!\, c_r!\, \underline{l}^r!} \sum_{\underline{j}}$$

$$\times \sum_{a=1}^{k_1+1} \left(\prod_{t \neq a} \frac{x_{j^1_a}}{x_{j^1_a} - x_{j^1_t}} \right) \cdot \frac{E^{b_1}_{x_{j^1_a}} e^{NQ(x_{j^1_a})}}{e^{NQ(x_{j^1_a})}}$$

$$\times \sum_{\substack{b=1 \\ j^1_a < j^2_b}}^{k_2+1} \left(\prod_{t \neq b} \frac{x_{j^2_b}}{x_{j^2_b} - x_{j^2_t}} \right) \frac{E^{b_2}_{x_{j^2_b}} e^{NQ(x_{j^2_b})}}{e^{NQ(x_{j^2_b})}} \frac{E^{l^1_0}_{x_{j^1_a}} E^{l^2_0}_{x_{j^2_b}} (x_{j^1_a} - x_{j^2_b})}{x_{j^1_a} - x_{j^2_b}}$$

$$\times \frac{E^{c_1}_{x_{j^1_a}} E^{c_2}_{x_{j^2_b}} T_N(\underline{x})}{T_N(\underline{x})}$$

$\boxed{\text{Lem 6.9}}$

$$\xrightarrow[x\to(1^N)]{} \sum_{a_r+b_r+c_r=q}\ \sum_{k_r}\ \sum_{\underline{\ell}^r}\ \prod_{r=1,2}\frac{q!}{b_r!\,c_r!\,\underline{\ell}^r!}\ \underbrace{\sum_{\underline{j}}}\ = \binom{N}{k_1+1}\binom{N}{k_2+1} \sim \frac{N^{k_1+k_2+2}}{(k_1+1)!\,(k_2+1)!}$$

$$\frac{1}{k_1!}\left(\frac{d}{dx}\right)^{k_1}\Big|_{x=1}\Bigg\{ x^{k_1}\frac{E_x^{b_1}e^{NQ(x)}}{e^{NQ(x)}}$$

$$\times\frac{1}{k_2!}\left(\frac{d}{dy}\right)^{k_2}\Big|_{y=1}\ y^{k_2}\frac{E_y^{b_2}e^{NQ(y)}}{e^{NQ(y)}}\ \frac{E_x^{\ell_0^1}E_y^{\ell_0^2}(x-y)}{x-y}\cdot\frac{E_x^{c_1}E_y^{c_2}T_N(x,y,1^{N-2})}{T_N(x,y,1^{N-2})}\Bigg\}$$

$$\boxed{c_r=0}$$

$$\sim \sum_{a_r+b_r=q}\ \sum_{k_r}\ \sum_{\underline{\ell}^r}\ \prod_{r=1,2}\frac{q!}{b_r!\,\underline{\ell}^r!}\ \frac{N^{k_r+1}}{(k_r+1)!}$$

$$\frac{1}{k_1!}\left(\frac{d}{dx}\right)^{k_1}\Big|_{x=1}\Bigg\{x^{k_1+b_1}N^{b_1}Q'(x)^{b_1}\cdot\frac{1}{k_2!}\left(\frac{d}{dy}\right)^{k_2}\Big|_{y=1}\ y^{k_2+b_2}N^{b_2}Q'(y)^{b_2}$$

$$\times\frac{E_x^{\ell_0^1}E_y^{\ell_0^2}(x-y)}{x-y}\Bigg\}$$

$$\boxed{k_r=a_r}\ \Rightarrow\ \boxed{\ell^r_t=1\ (t\geq 1),\quad \ell^r_0=0}$$

$$\sim\sum_{a_r+b_r=q}\ \prod_{r=1,2}\frac{q!\,N^{a_1+b_1+1}}{b_r!\,(a_r+1)!}\quad\frac{1}{a_1!}\left(\frac{d}{dx}\right)^{a_1}\Big|_{x=1}\left(x^{a_1+b_1}Q'(x)^{b_1}\right)$$

$$\times\frac{1}{a_2}\left(\frac{d}{dy}\right)^{a_2}\Big|_{y=1}\left(y^{a_2+b_2}Q'(x)^{b_2}\right)$$

$$= N^{2(q+1)}\left(\sum_{a=0}^{q}\frac{q!}{(q-a)!\,(a+1)!\,a!}\left(\frac{d}{dx}\right)^{a}\Big|_{x=1}\left(x^{q}\,Q'(x)^{q-a}\right)\right)^2$$